RELATIVE GOURSAT CATEGORIES

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Abstract. We define relative Goursat categories and prove relative versions of the equivalent conditions defining regular Goursat categories. These include 3-permutability of equivalence relations, preservation of equivalence relations under direct images, a condition on so-called Goursat pushouts, and the denormalised 3×3 Lemma. This extends recent work by Gran and Rodelo on a new characterisation of Goursat categories to a relative context.

Introduction

According to A. Carboni, G. M. Kelly and M. C. Pedicchio [3], a **Goursat category** can be defined as a regular category satisfying the 3-permutability of equivalence relations, that is, having $RSR = SRS$ for every two equivalence relations R and S on the same object. However, it is known that there are several other equivalent definitions and characterisations, including the following two, recently obtained by M. Gran and D. Rodelo [5]:

1. A regular category $\mathcal{A}$ is Goursat if and only if for every (downward) split epimorphism of regular epimorphims

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\downarrow\uparrow & & {\scriptstyle g}\downarrow\uparrow \\ B & \longrightarrow & D \end{array}$$

   in $\mathcal{A}$ (that is, f and g are compatibly split), the induced morphism between the kernel pairs of f and g is a regular epimorphism; such squares are called **Goursat pushouts** (they are automatically pushouts).
2. A regular category $\mathcal{A}$ is Goursat if and only if it satisfies the so-called *denormalised 3×3 Lemma*.

This denormalised 3×3 Lemma was first introduced and proved in a regular Mal'tsev context by D. Bourn in [2] and was proved to hold also in regular Goursat categories by S. Lack in [11].

The purpose of the present paper is to extend these two results to characterise what we call **relative Goursat categories** by making the following replacements:

· The regular category $\mathcal{A}$ is replaced with a pair $(\mathcal{A}, \mathcal{E})$ where $\mathcal{E}$ is a class of regular epimorphisms in $\mathcal{A}$ satisfying suitable conditions; when $\mathcal{A}$ has all finite limits and coequalisers of kernel pairs and $\mathcal{E}$ is the class of all regular epimorphisms in $\mathcal{A}$, these conditions make $\mathcal{A}$ regular;
· the Goursat pushouts are required to have all their regular epimorphisms in $\mathcal{E}$;
· the 3×3 Lemma is replaced by its *$\mathcal{E}$-relative version* (see Theorem 3.3 below).

In fact, we show in detail that our conditions on $(\mathcal{A}, \mathcal{E})$ allow us to repeat essentially all arguments of [11] and [5]. Our main tool here is the calculus of $\mathcal{E}$-relations developed in [7, 8], which does not require $\mathcal{E}$ to be part of a factorisation system. Its original motivation was to introduce and study relative semi-abelian and relative homological categories in [8, 7, 6]. We can link these different settings by the following chain of implications:

$$\text{relative semi-abelian [8, 7]} \Rightarrow \text{relative homological [6]} \Rightarrow$$
$$\text{relative Mal'tsev [4]} \Rightarrow \text{relative Goursat} \Rightarrow \text{relative regular.}$$

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1. Relations in a relative setting

When working with relations, one usually uses the regular image factorisation in a regular category to obtain composition of relations (see e.g. [3]). Recall that a **regular category** is a finitely complete category with coequalisers of kernel pairs and pullback-stable regular epimorphisms [1]. In a relative setting, regular epimorphisms are replaced with a suitable class $\mathcal{E}$ of regular epimorphisms in the ground category $\mathcal{A}$, and a relative factorisation axiom is used instead of regular image factorisation to compose relations [8] (see also [7]).

In this paper, we consider a slightly more general setting for relations than the one in [8], namely, we do not require the existence of all pullbacks, but only ask for pullbacks of morphisms in $\mathcal{E}$ to exist. We introduce:

Definition 1.1. A **relative regular category** is a pair $(\mathcal{A}, \mathcal{E})$ where $\mathcal{A}$ is a category with finite products and $\mathcal{E}$ is a class of regular epimorphisms in $\mathcal{A}$ such that the following axioms hold:

- (E1) $\mathcal{E}$ contains all isomorphisms;
- (E2) pullbacks of morphisms in $\mathcal{E}$ exist in $\mathcal{A}$ and are in $\mathcal{E}$;
- (E3) $\mathcal{E}$ is closed under composition;
- (E4) if $f \in \mathcal{E}$ and $gf \in \mathcal{E}$ then $g \in \mathcal{E}$;
- (F) if a morphism f in $\mathcal{A}$ factors as $f = em$ with m a monomorphism and $e \in \mathcal{E}$, then it also factors (essentially uniquely) as $f = m'e'$ with m' a monomorphism and $e' \in \mathcal{E}$.

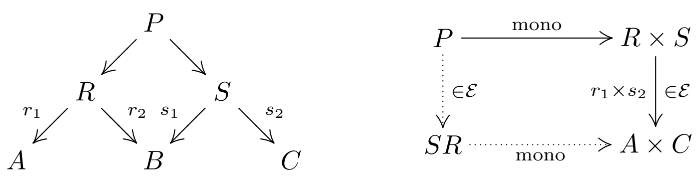
Note that if $\mathcal{A}$ is a category with products and all pullbacks and $\mathcal{E}$ is a class of regular epimorphisms in $\mathcal{A}$ containing all isomorphisms, then $(\mathcal{A}, \mathcal{E})$ is a relative regular category if and only if $(\mathcal{A}, \mathcal{E})$ satisfies Condition 2.1 of [8]. Note also that this context is not as general as the one considered in [9] where products are not required to exist. Therefore, the level of generality here is between those of [8] and [9].

Remark 1.2 (The "absolute case")**.** As easily follows from Definition 1.1, if $\mathcal{A}$ is a category with finite limits and has coequalisers of kernel pairs and $\mathcal{E}$ is the class of all regular epimorphisms in $\mathcal{A}$, then $(\mathcal{A}, \mathcal{E})$ is a relative regular category if and only if $\mathcal{A}$ is a regular category.

Relative regular categories provide a convenient setting for the calculus of $\mathcal{E}$-*relations* in the same way that regular categories do for the calculus of relations. The following definitions and properties of $\mathcal{E}$-relations are the relative versions of classical properties of relations. Their proofs easily follow those of the absolute version and appear in [7] (the absolute versions can be found for example in [3]).

Definition 1.3. Given two objects A and B in $\mathcal{A}$, an $\mathcal{E}$**-relation** R **from** A **to** B is a subobject $\langle r_1, r_2\rangle \colon R \to A \times B$ of $A \times B$ such that the morphisms $r_1 \colon R \to A$ and $r_2 \colon R \to B$ are in $\mathcal{E}$. We denote such an $\mathcal{E}$-relation by (R, r_1, r_2) or just by R, and its **opposite** (R, r_2, r_1) by R°; we will also write $R \colon A \to B$ for an $\mathcal{E}$-relation R from A to B. When $A = B$, we say that R is an $\mathcal{E}$**-relation on** A or an **endo-**$\mathcal{E}$**-relation**.

We can compose two $\mathcal{E}$-relations (R, r_1, r_2) from A to B and (S, s_1, s_2) from B to C by forming the pullback of r_2 and s_1 and then using the factorisation from Axiom (F) to obtain a monomorphism $SR \to A \times C$:



Axioms (E2), (E3) and (E4) ensure that this composite is again an $\mathcal{E}$-relation. Moreover, the composition is associative (as we identify isomorphic relations) and we have:

- $(R^\circ)^\circ = R$,
- $(SR)^\circ = R^\circ S^\circ$,
- $R \leq R' \Rightarrow R^\circ \leq R'^\circ$ (ordering as subobjects),
- if $R \leq R'$ and $S \leq S'$ then $SR \leq S'R'$,

for all $\mathcal{E}$-relations $R \colon A \to B$, $R' \colon A \to B$, $S \colon B \to C$, and $S' \colon B \to C$ in $\mathcal{A}$.

Remark 1.4. Given a morphism $f \colon A \to B$ in $\mathcal{E}$, we can use (E1) to view f as an $\mathcal{E}$-relation $f = (A, 1_A, f)$; its opposite is $f^\circ = (A, f, 1_A)$. It is easy to see (cf. [3, 8]) that

- $f^\circ f$ is the kernel pair of f,

- $ff^\circ = 1_B$,
- $ff^\circ f = f$,
- $f^\circ ff^\circ = f^\circ$,
- for any $\mathcal{E}$-relation (R, r_1, r_2) we have $R = r_2 r_1^\circ$.

Definition 1.5. An $\mathcal{E}$-relation (R, r_1, r_2) on an object A in $\mathcal{A}$ is said to be

- **reflexive** if $1_A \leq R$,
- **symmetric** if $R^\circ \leq R$ (and thus $R^\circ = R$),
- **transitive** if $RR \leq R$,
- an **equivalence $\mathcal{E}$-relation** if it is reflexive, symmetric and transitive;
- an **$\mathcal{E}$-effective equivalence $\mathcal{E}$-relation** if it is a kernel pair of some morphism in $\mathcal{E}$.

As easily follows from Definition 1.5, an $\mathcal{E}$-relation $R\colon A \to A$ which is reflexive and transitive satisfies $RR = R$. Note also that the kernel pair of any morphism $f \in \mathcal{E}$ is an ($\mathcal{E}$-effective) equivalence $\mathcal{E}$-relation, by pullback-stability (E2).

This allows us to copy the $n = 3$ case of [3, Theorem 3.5] to a relative version. We give the proof for convenience.

Proposition 1.6. *Let $(\mathcal{A}, \mathcal{E})$ be a relative regular category. The following conditions are equivalent:*

(i) *for equivalence $\mathcal{E}$-relations R and S on an object A, we have $RSR = SRS$;*
(ii) *this 3-permutability $RSR = SRS$ holds when R and S are $\mathcal{E}$-effective equivalence $\mathcal{E}$-relations;*
(iii) *every $\mathcal{E}$-relation P from A to B satisfies $PP^\circ PP^\circ = PP^\circ$;*
(iv) *for every reflexive $\mathcal{E}$-relation E on an object A, the $\mathcal{E}$-relation EE° is an equivalence $\mathcal{E}$-relation;*
(v) *for every reflexive $\mathcal{E}$-relation E, the $\mathcal{E}$-relation EE° is transitive;*
(vi) *for every reflexive $\mathcal{E}$-relation E, we have $EE^\circ = E^\circ E$.*

Proof. Clearly (i) $\Rightarrow$ (ii). Given an $\mathcal{E}$-relation P from A to B we view it as $\langle p_1, p_2\rangle\colon P \to A \times B$ such that $P = p_2 p_1^\circ$. Then $p_1^\circ p_1$ and $p_2^\circ p_2$ are the kernel pairs of p_1 and p_2 respectively, and therefore $\mathcal{E}$-effective equivalence $\mathcal{E}$-relations. Hence, by (ii) and using Remark 1.4, we obtain:

$$PP^\circ PP^\circ = p_2 p_1^\circ p_1 p_2^\circ p_2 p_1^\circ p_1 p_2^\circ = p_2 p_2^\circ p_2 p_1^\circ p_1 p_2^\circ p_2 p_2^\circ = p_2 p_1^\circ p_1 p_2^\circ = PP^\circ,$$

proving (ii) $\Rightarrow$ (iii). Now given a reflexive $\mathcal{E}$-relation E on A, the reflexivity $1_A \leq E$ and the induced $1_A \leq E^\circ$ imply $1_A \leq EE^\circ$, giving reflexivity of EE°. Symmetry is automatic as $(EE^\circ)^\circ = EE^\circ$, and transitivity $EE^\circ EE^\circ = EE^\circ$ follows from (iii), therefore (iii) $\Rightarrow$ (iv). Clearly (iv) $\Rightarrow$ (v), and (v) $\Rightarrow$ (vi) since reflexivity of E gives

$$E^\circ E \leq EE^\circ EE^\circ \leq EE^\circ.$$

It remains to prove (vi) $\Rightarrow$ (i). Given two equivalence $\mathcal{E}$-relations R and S on an object A, we have $R = R^\circ$, $RR^\circ = R$ and the same for S; moreover, their composite $E = SR$ is clearly reflexive. Therefore we have

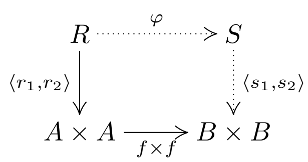
$$SRS = SRR^\circ S^\circ = R^\circ S^\circ SR = RSR,$$

which gives (i). □

The case $n = 2$ of [3, Theorem 3.5] defines a regular Mal'tsev category and is stated in its relative version in [4].

In the absolute case of regular categories, it is possible to form the direct image of any endo-relation [3]. In a similar way, we can form an $\mathcal{E}$-image of an endo-$\mathcal{E}$-relation in our relative setting.

Definition 1.7. Let $(\mathcal{A}, \mathcal{E})$ be a relative regular category. Given an $\mathcal{E}$-relation (R, r_1, r_2) on an object A of $\mathcal{A}$ and a morphism $f\colon A \to B$ in $\mathcal{E}$, we define the **$\mathcal{E}$-image of R along f** to be the relation S on B which is induced by the ($\mathcal{E}$, mono)-factorisation $\langle s_1, s_2\rangle\varphi$ of the morphism $(f \times f)\langle r_1, r_2\rangle$



which exists by axiom (F). We write $f(R) = S$, which is again an $\mathcal{E}$-relation by axiom (E4).

Remark 1.8. The essential uniqueness of ($\mathcal{E}$, mono)-factorisations implies that, when R is a reflexive $\mathcal{E}$-relation, then $f(R)$ is also reflexive, and when R is a symmetric $\mathcal{E}$-relation, then $f(R)$ is symmetric. In the next section we will see under which conditions the $\mathcal{E}$-image $f(R)$ of an equivalence $\mathcal{E}$-relation R is again an equivalence $\mathcal{E}$-relation.

As in the absolute case [3], we have an easy way to form the $\mathcal{E}$-image:

Lemma 1.9. *Let $(\mathcal{A}, \mathcal{E})$ be a relative regular category. Given an $\mathcal{E}$-relation (R, r_1, r_2) on an object A in $\mathcal{A}$ and a morphism $f\colon A \to B$ in $\mathcal{E}$, the $\mathcal{E}$-image $f(R)$ can be formed as the composite $f(R) = fRf^{\circ} = fr_2r_1^{\circ}f^{\circ}$.* □

Furthermore, using Remark 1.4 and the definition of $\mathcal{E}$-image as well as Lemma 1.9, we easily see the following.

Corollary 1.10. *Let $(\mathcal{A}, \mathcal{E})$ be a relative regular category. Given a commutative diagram*

$$\begin{array}{ccc} R & \underset{r_2}{\overset{r_1}{\rightrightarrows}} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ S & \underset{s_2}{\overset{s_1}{\rightrightarrows}} & B \end{array}$$

where R and S are $\mathcal{E}$-relations and $f \in \mathcal{E}$, the morphism g is in $\mathcal{E}$ if and only if $S = f(R)$, or equivalently if and only if $s_2s_1^{\circ} = fr_2r_1^{\circ}f^{\circ}$. If (R, r_1, r_2) and (S, s_1, s_2) are kernel pairs with coequalisers r and s, respectively, in $\mathcal{E}$, then the latter is also equivalent to $s^{\circ}s = fr^{\circ}rf^{\circ}$. □

For the main result in the next section, we need the following lemma (c.f. [4]):

Lemma 1.11. *Let $(\mathcal{A}, \mathcal{E})$ be a relative regular category. Given a morphism of (downward) split epimorphisms*

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow\uparrow & & {\scriptstyle g}\downarrow\uparrow \\ B & \xrightarrow[k]{} & D \end{array}$$

with f in $\mathcal{E}$, the induced morphism between the kernel pairs of h and k is also in $\mathcal{E}$.

Proof. Writing (H, h_1, h_2) and (K, k_1, k_2) for the kernel pairs of h and k respectively, clearly the induced morphism $H \to K$ is again a split epimorphism. Axiom (F) allows us to factorise the morphism

$$(f \times f)\langle h_1, h_2\rangle \colon H \to B \times B$$

as a morphism $e \in \mathcal{E}$ followed by a relation $\langle r_1, r_2\rangle \colon R \to B \times B$ on B.

$$\begin{array}{ccc} H & \xrightarrow{\langle h_1, h_2\rangle} & A \times A \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle f \times f} \\ R & \xrightarrow[\langle r_1, r_2\rangle]{} & B \times B \end{array}$$

Since e is in particular an epimorphism, the relation R factors over the kernel pair K of k. But since $H \to K$ is a split epimorphism, we see that $R \cong K$ and thus the induced morphism $H \to K$ is in $\mathcal{E}$. □

2. The relative Goursat axiom

We now prove an equivalence of several conditions, which in the absolute case all characterise regular Goursat categories (see [3] and [5]).

Theorem 2.1. *Let $(\mathcal{A}, \mathcal{E})$ be a relative regular category. Then the following conditions are equivalent:*

(i) *the **$\mathcal{E}$-Goursat axiom** holds: given a morphism of (downward) split epimorphisms*

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow\uparrow & & {\scriptstyle g}\downarrow\uparrow \\ B & \xrightarrow[k]{} & D \end{array} \qquad (1)$$

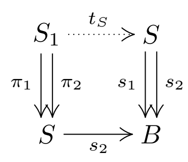
in $\mathcal{A}$ with f, g, h and k in $\mathcal{E}$, the induced morphism between the kernel pairs of f and g is also in $\mathcal{E}$;

(ii) *the $\mathcal{E}$-image of an equivalence $\mathcal{E}$-relation is an equivalence $\mathcal{E}$-relation;*

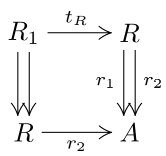
(iii) *for every reflexive $\mathcal{E}$-relation E on an object A, the $\mathcal{E}$-relation EE° is an equivalence $\mathcal{E}$-relation;*

(iv) *for equivalence $\mathcal{E}$-relations R and S on an object A, we have $RSR = SRS$.*

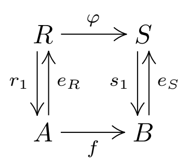
Proof. The proof of (i) $\Rightarrow$ (ii) is the same as its absolute version given in [5, Theorem 2.3]. We give it here for completeness. Let (R, r_1, r_2) be an equivalence $\mathcal{E}$-relation on A and let $f\colon A \to B$ be in $\mathcal{E}$. We want to show that the $\mathcal{E}$-image $f(R) = (S, s_1, s_2)$ of R along f is again an equivalence $\mathcal{E}$-relation. Since S is reflexive and symmetric by Remark 1.8, we only have to show that it is transitive, that is, $SS \leq S$. However, since S is symmetric, it suffices to show the existence of a morphism $t_S\colon S_1 \to S$, where (S_1, π_1, π_2) is the kernel pair of s_1, which makes the following diagram commute:



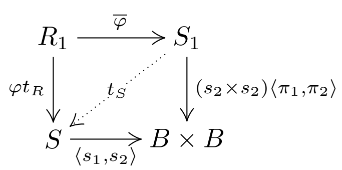
Since R is (symmetric and) transitive, there exists a morphism $t_R\colon R_1 \to R$, where R_1 is the kernel pair of r_1, making the corresponding diagram for R commute:



Using the morphisms e_R and e_S which define the reflexivity of R and S, we obtain a diagram



of type (**1**), where φ is the $\mathcal{E}$-part of the ($\mathcal{E}$, mono)-factorisation $\langle s_1, s_2\rangle\varphi$ of $(f \times f)\langle r_1, r_2\rangle$. Therefore, by (i), the induced morphism $\overline{\varphi}\colon R_1 \to S_1$ between the kernel pairs of r_1 and s_1 is in $\mathcal{E}$. Since every morphism in $\mathcal{E}$ is a regular epimorphism and therefore a strong epimorphism and $\langle s_1, s_2\rangle$ is a monomorphism, we obtain a unique diagonal t_S in the square



making both triangles commute, which is the required morphism.

For (ii) $\Rightarrow$ (iii) it is easy to see that for a reflexive $\mathcal{E}$-relation (E, e_1, e_2) on an object A we have $EE^\circ = e_2(E_1)$, where E_1 is the kernel pair of e_1. Therefore EE° is an equivalence $\mathcal{E}$-relation as an $\mathcal{E}$-image of the equivalence $\mathcal{E}$-relation E_1. Conditions (iii) and (iv) are equivalent by Proposition 1.6. Finally, for (iv) $\Rightarrow$ (i) we again use the proof from [5, Theorem 2.3]. For convenience, we copy the proof and add our adapted justifications for the relative setting.

Given a diagram such as (**1**), Lemma 1.11 implies that the induced split epimorphism between the kernel pairs H of h and K of k is in $\mathcal{E}$. This means that $f(H) = K$. Now using Lemma 1.9 and the three-permutability from (iv) on the kernel pairs $H = h^\circ h$ and $F = f^\circ f$, we see that

$$\begin{aligned} h(F) &= hf^\circ fh^\circ && \text{(by Lemma 1.9)} \\ &= hh^\circ hf^\circ fh^\circ hh^\circ && \text{(since } hh^\circ h = h) \\ &= hf^\circ fh^\circ hf^\circ fh^\circ && \text{(by (iv))} \\ &= hf^\circ k^\circ kfh^\circ && \text{(since } f(H) = K) \\ &= hh^\circ g^\circ ghh^\circ && \text{(since } kf = gh) \\ &= g^\circ g && \text{(since } hh^\circ = 1) \\ &= G, \end{aligned}$$

where G is the kernel pair of g. By Corollary 1.10, this implies that the induced morphism between the kernel pairs $F \to G$ is in $\mathcal{E}$. □

Note that since all morphisms in $\mathcal{E}$ are in particular epimorphisms, the $\mathcal{E}$-Goursat axiom implies that every such square (**1**) is a pushout, giving the relative notion of an **$\mathcal{E}$-Goursat pushout**. Since all the conditions above characterise Goursat categories in the absolute case [3, 5], we are now justified in the following definition.

Definition 2.2. A **relative Goursat category** is a relative regular category $(\mathcal{A}, \mathcal{E})$ in which moreover the following axiom holds:

(G) the **$\mathcal{E}$-Goursat axiom**: given a morphism of (downward) split epimorphisms

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ f \downarrow\uparrow & & g \downarrow\uparrow \\ B & \xrightarrow[k]{} & D \end{array}$$

in $\mathcal{A}$ with f, g, h and k in $\mathcal{E}$, the induced morphism between the kernel pairs of f and g is also in $\mathcal{E}$.

Compare this to the definition of relative Mal'tsev categories in [4]: the *$\mathcal{E}$-Mal'tsev axiom* (E5) given there says that for any morphism of split epimorphisms (**1**) with f, g, h and k in $\mathcal{E}$, the canonical morphism $\langle f, h\rangle$ to the pullback $B \times_D C$ is also in $\mathcal{E}$. As in the absolute case, this relative Mal'tsev axiom implies the $\mathcal{E}$-Goursat axiom (G). In [4] this is proved via pullback-stability of so-called double extensions, that is, squares of extensions where the comparison morphism to the pullback is also an extension. The implication can also be seen easily via the properties of relations: if for every two equivalence $\mathcal{E}$-relations R and S on an object A we have $RS = SR$ and this is an equivalence relation, then also $RSR = SRS$.

3. The relative 3×3 Lemma

In this section we prove the relative version of the so-called denormalised 3×3 Lemma in the context of relative Goursat categories. Furthermore, following the absolute case laid out in [5], we show that, in a relative regular category $(\mathcal{A}, \mathcal{E})$, the relative 3×3 Lemma is in fact equivalent to the $\mathcal{E}$-Goursat axiom.

Definition 3.1. Let $(\mathcal{A}, \mathcal{E})$ be a relative regular category. We will say that the diagram

$$F \underset{f_2}{\overset{f_1}{\rightrightarrows}} A \xrightarrow{f} B \qquad (\mathbf{2})$$

is **$\mathcal{E}$-exact** when (F, f_1, f_2) is the kernel pair of f and f is in $\mathcal{E}$.

Note that when (**2**) is $\mathcal{E}$-exact, the morphisms f_1 and f_2 are also in $\mathcal{E}$ by pullback-stability (E2).

In the proof of the relative 3×3 Lemma, we will need the following

Lemma 3.2. [8, Theorem 2.10] *Let $(\mathcal{A}, \mathcal{E})$ be a relative regular category. Given a diagram*

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ f \downarrow & & \downarrow g \\ B & \xrightarrow[k]{} & D \end{array}$$

with the morphisms f, h, k, and g in $\mathcal{E}$, we have $hf^\circ = g^\circ k$ if and only if $kf = gh$ and the canonical morphism $\langle f, h\rangle\colon A \to B \times_D C$ is in $\mathcal{E}$. □

We first show that the relative 3×3 Lemma does indeed hold in any relative Goursat category.

Theorem 3.3 (The relative 3×3 Lemma)**.** *Let $(\mathcal{A}, \mathcal{E})$ be a relative Goursat category. Given a commutative diagram*

$$\begin{array}{ccccc}
\overline{F} & \underset{\overline{h}_2}{\overset{\overline{h}_1}{\rightrightarrows}} & F & \xrightarrow{\overline{h}} & G \\
{\scriptstyle \overline{f}_2}\downdownarrows{\scriptstyle \overline{f}_1} & & {\scriptstyle f_2}\downdownarrows{\scriptstyle f_1} & & {\scriptstyle g_2}\downdownarrows{\scriptstyle g_1} \\
H & \underset{h_2}{\overset{h_1}{\rightrightarrows}} & A & \xrightarrow{h} & C \\
{\scriptstyle \overline{f}}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
K & \underset{k_2}{\overset{k_1}{\rightrightarrows}} & B & \xrightarrow{k} & D,
\end{array} \qquad (\mathbf{3})$$

satisfying the usual commutativity conditions (such as $f_i\overline{h}_j = h_j\overline{f}_i$), and with $\mathcal{E}$-exact columns and second row, the first row is $\mathcal{E}$-exact if and only if the third row is $\mathcal{E}$-exact.

Proof. The proof is the same as in the absolute case [11]; we repeat it here with the appropriate justifications for the relative case.

Suppose the third row is $\mathcal{E}$-exact. Since k_1 and k_2 are jointly monic, an easy diagram chase proves that $(\overline{F}, \overline{h_1}, \overline{h_2})$ is the kernel pair of $\overline{h}$. Therefore, it remains to show that $\overline{h}$ is in $\mathcal{E}$. Note that since $\overline{f}$ is in $\mathcal{E}$, Corollary 1.10 implies that the $\mathcal{E}$-image $f(H) = fHf^\circ$ is equal to K. We have:

$$\begin{aligned}
g^\circ g &= hh^\circ g^\circ ghh^\circ && \text{(since } hh^\circ = 1)\\
&= hf^\circ k^\circ kfh^\circ && \text{(since } gh = kf)\\
&= hf^\circ fh^\circ hf^\circ fh^\circ && \text{(since } f(H) = K)\\
&= hh^\circ hf^\circ fh^\circ hh^\circ && \text{(by Theorem 2.1(iv))}\\
&= hf^\circ fh^\circ && \text{(since } hh^\circ h = h)
\end{aligned}$$

Therefore $G = h(F)$ and, by Corollary 1.10, $\overline{h}$ is in $\mathcal{E}$.

Conversely, suppose the first row is $\mathcal{E}$-exact. By Axiom (E4) the morphisms k_1, k_2 and k are in $\mathcal{E}$. We have:

$$\begin{aligned}
k^\circ k &= ff^\circ k^\circ kff^\circ && \text{(since } ff^\circ = 1)\\
&= fh^\circ g^\circ ghf^\circ && \text{(since } gh = kf)\\
&= fh^\circ hf^\circ fh^\circ hf^\circ && \text{(since } h(F) = G)\\
&= ff^\circ fh^\circ hf^\circ ff^\circ && \text{(by Theorem 2.1(iv))}\\
&= fh^\circ hf^\circ && \text{(since } ff^\circ f = f)\\
&= fh_2h_1^\circ f^\circ && \text{(since } H = h^\circ h = h_2h_1^\circ)\\
&= k_1\overline{f}\overline{f}^\circ k_2^\circ && \text{(since } fh_i = k_i\overline{f})\\
&= k_1k_2^\circ && \text{(since } \overline{f}\overline{f}^\circ = 1)
\end{aligned}$$

Therefore, since $k^\circ k = k_1k_2^\circ$ and the morphisms k_1, k_2, and k are in $\mathcal{E}$, Lemma 3.2 implies that $k_1'e = k_1$ and $k_2'e = k_2$ for some $e \in \mathcal{E}$, where (k_1', k_2') is the kernel pair of k. It remains to prove that e is an isomorphism. For this, we replace (k_1, k_2) by (k_1', k_2') and $\overline{f}$ by $e\overline{f}$ in the 3×3 diagram (**3**). Then the three rows, the second column and the third column are $\mathcal{E}$-exact; therefore, by the first part of the proof, the first column is also $\mathcal{E}$-exact. Thus $\overline{f}$ and $e\overline{f}$ are both coequalisers of $\overline{f}_1$ and $\overline{f}_2$, yielding that e is an isomorphism. □

Now we show that the relative 3×3 Lemma is equivalent to the $\mathcal{E}$-Goursat axiom in a relative regular category.

Theorem 3.4. *Let $(\mathcal{A}, \mathcal{E})$ be a relative regular category. The following conditions are equivalent:*

(i) *$(\mathcal{A}, \mathcal{E})$ is a relative Goursat category, that is, Axiom (G) holds in $(\mathcal{A}, \mathcal{E})$;*

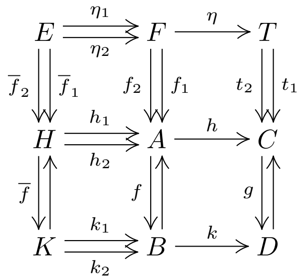
(ii) *the relative 3×3 Lemma holds in $(\mathcal{A}, \mathcal{E})$;*

(iii) *in a diagram such as* (**3**), *if the first row is* $\mathcal{E}$*-exact then the third row is also* $\mathcal{E}$*-exact;*
(iv) *in a diagram such as* (**3**), *if the third row is* $\mathcal{E}$*-exact then the first row is also* $\mathcal{E}$*-exact.*

Proof. The proof is the same as in the absolute case [5]; we again repeat it for convenience. The part (i) $\Rightarrow$ (ii) is Theorem 3.3, and the implications (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (i): Consider the commutative diagram (**1**) with the assumptions of axiom (G). Let (H, h_1, h_2) and (K, k_1, k_2) be the kernel pairs of h and k respectively, and let $\overline{f}\colon H \to K$ be the induced morphism; since f is in $\mathcal{E}$, Lemma 1.11 implies that $\overline{f}$ is also in $\mathcal{E}$. Taking the kernel pairs of $\overline{f}$, f and g and the induced morphisms between them, we obtain the diagram (**3**) with the three columns and the second and the third rows $\mathcal{E}$-exact. Then (iv) implies that the first row is also $\mathcal{E}$-exact, and therefore the induced morphism between the kernel pairs of f and g is in $\mathcal{E}$, proving (i).

(iii) $\Rightarrow$ (i): Again consider the commutative diagram (**1**). Let (F, f_1, f_2) be the kernel pair of f and let (T, t_1, t_2) be the $\mathcal{E}$-image of F along h. To prove (i) it suffices to show that (T, t_1, t_2) is the kernel pair of g. For this, consider the commutative diagram



(**4**)

in which (E, η_1, η_2), (H, h_1, h_2), and (K, k_1, k_2) are the kernel pairs of η, h, and k respectively, and $\overline{f}_1, \overline{f}_2\colon E \to H$ are the induced morphisms. Since f is in $\mathcal{E}$, Lemma 1.11 implies that $\overline{f}$ is also in $\mathcal{E}$. Moreover, since t_1 and t_2 are jointly monic, it follows that $(E, \overline{f}_1, \overline{f}_2)$ is the kernel pair of $\overline{f}$, making the first column $\mathcal{E}$-exact. Then, applying (iii) to Diagram (**4**) with the role of columns and rows interchanged, we obtain that the third column is also $\mathcal{E}$-exact. Therefore, (T, t_1, t_2) is the kernel pair of g, as desired. ☐

Remark 3.5. Theorem 3.4 is in fact the relative version of [5, Proposition 3.2]; in particular, our (iii) $\Rightarrow$ (iv) corresponds to 3.2(c) $\Rightarrow$ 3.2(d) in [5], which itself can be considered as a non-pointed version of 5.4(b) $\Rightarrow$ 5.4(c) in [10].

As mentioned in the introduction, we have a chain of implications of various relative categories: any relative Goursat category is by definition relatively regular, and any relative Mal'tsev category is relatively Goursat, as mentioned above. Furthermore, every relative homological category is relatively Mal'tsev by [8, Theorem 2.14] (this is shown in a different way, not using relations, in [4]), and any relative semi-abelian category is relatively homological by definition, see [8, Definition 3.2]. This is the same chain of implications as in the absolute case.

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