

The fundamental group functor as a Kan extension

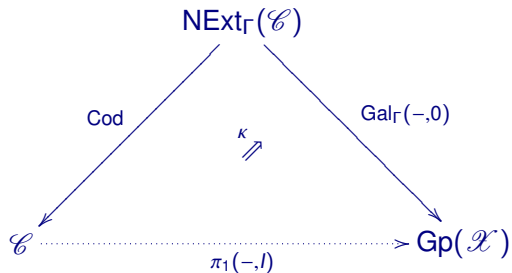
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University of Cambridge

joint work with Tomas Everaert and Tim Van der Linden

25 March 2013, PSSL Sheffield

Aim of the talk



Why?

- In topological example gives some universal properties of the “usual” fundamental group, and (hopefully) also the connecting homomorphism in exact sequence induced by a fibration.
- In algebraic examples gives another approach to semi-abelian homology.

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Galois structures

Definition (Janelidze)

A **Galois structure** Γ consists of an adjunction

$$\mathcal{C} \begin{array}{c} \xrightarrow{I} \\ \perp \\ \xleftarrow{H} \end{array} \mathcal{X}$$

with unit η and counit ϵ , as well as classes of maps \mathcal{E} in \mathcal{C} and \mathcal{F} in \mathcal{X} satisfying certain axioms.

Examples of Galois structures

- Groups with subcategory abelian groups, regular epis.
- Semi-abelian \mathcal{C} with Birkhoff subcategory, regular epis.
- Locally connected topological spaces and sets.
 H is discrete topology functor,
 $I = \pi_0$, connected components functor.
 \mathcal{E} = local homeomorphisms (étale maps), \mathcal{F} = all maps.
- Opposite of finite dimensional k -algebras, finite sets, each with all maps. The adjunction is defined through idempotent decomposition of k -algebras: a k -algebra is sent to its set of primitive idempotents.

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Special maps in Galois structures

- trivial coverings:

$$\begin{array}{ccc}
 A & \xrightarrow{\eta_A} & HIA \\
 \mathcal{E} \ni f \downarrow & \lrcorner & \downarrow HIf \\
 B & \xrightarrow{\eta_B} & HIB
 \end{array}$$

(cartesian wrt. I)

- monadic extensions: $p: E \rightarrow B$ in \mathcal{E} with $p^*: (\mathcal{E} \downarrow B) \rightarrow (\mathcal{E} \downarrow E)$ monadic.
(good to pull back along)
- coverings (or central extensions): $f \in \mathcal{E}$ with $p^*(f)$ trivial for some monadic p . (locally trivial)
- normal extensions: monadic p with trivial kernel pair projections.

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 - monadic extensions: all regular epis.
 - central extensions as usual, kernel inside centre;
- topological example:
 - monadic extensions: surjective étale maps;
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Admissibility

From now on H is inclusion.

- **admissible Galois structure**: I preserves pullbacks along trivial coverings.
- When \mathcal{E} is all maps, admissible = semi-left exact = Street fibration.
- \Rightarrow Trivial coverings are pullback-stable.
- Think “trivial coverings are pullback-closure of \mathcal{F} in \mathcal{C} ”.
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Galois groupoid (Janelidze)

Take \mathcal{C} pointed and $p: E \rightarrow B$ normal extension.

- Galois groupoid $\text{Gal}_\Gamma(p) = \text{IEq}(p)$

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 \text{IEq}(p) & \begin{array}{c} \xrightarrow{Id} \\ \xrightarrow{Ic} \end{array} & IE & &
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Properties of Galois group functor

Morphisms in $\text{NExt}_\Gamma(\mathcal{C})$

$$\begin{array}{ccc} E & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & E' \\ p \downarrow & & \downarrow p' \\ B & \xrightarrow{b} & B' \end{array}$$

induce

- homotopy on kernel pairs and Galois groupoids,
- same morphism $\text{Gal}_\Gamma(p, 0) \rightarrow \text{Gal}_\Gamma(p', 0)$ on Galois group.
- $(f, 1_B): p \rightarrow p$ induces identity on Galois group.

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Fundamental group functor

Now assume that weakly universal normal extensions exist.

$$\begin{array}{ccc}
 & \text{NExt}_{\Gamma}(\mathcal{C}) & \\
 \text{Cod} \swarrow & & \searrow \text{Gal}_{\Gamma}(-, 0) \\
 \mathcal{C} & \xrightarrow{\kappa} & \text{Gp}(\mathcal{X}) \\
 \text{---} \xrightarrow{\pi_1(-, l)} & & \text{---}
 \end{array}$$

Definition (Janelidze)

Given $B \in \mathcal{C}$, pick weakly universal normal extension $u: U \rightarrow B$, and let

$$\pi_1(B, l) = \text{Gal}_{\Gamma}(u, 0).$$

Functorial in B because of induced homotopies.

Examples

- Groups and abelian groups: get $\pi_1(B, I) = H_2(B, \mathbb{Z})$ (group homology).
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Natural transformation κ

$$\kappa: \pi_1(-, I) \circ \text{Cod} \implies \text{Gal}_\Gamma(-, 0)$$

has components

$$\kappa_p: \pi_1(B, I) = \text{Gal}_\Gamma(u, 0) \longrightarrow \text{Gal}_\Gamma(p, 0)$$

for normal extension $p: E \longrightarrow B$, induced by (any)

$$\begin{array}{ccc} U & \xrightarrow{h} & E \\ u \downarrow & & \downarrow p \\ B & \xlongequal{\quad} & B \end{array}$$

That is, $\kappa_p = \text{Gal}_\Gamma((h, 1_B), 0)$.

Universality of κ

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define $\alpha: F \Rightarrow \pi_1(-, I)$ by $\alpha_B = \gamma_u: FB \rightarrow \pi_1(B, I)$.

Then

- α is natural by naturality of γ ;
- $\kappa_p \circ \alpha_{\text{Cod } p} = \gamma_p$ for all normal extensions p , by naturality of γ ;
- α is unique: given β with $\kappa_p \circ \beta_{\text{Cod } p} = \gamma_p$ for all normal p , get $\alpha_B = \beta_B$ as κ_u is an iso for weakly universal u .

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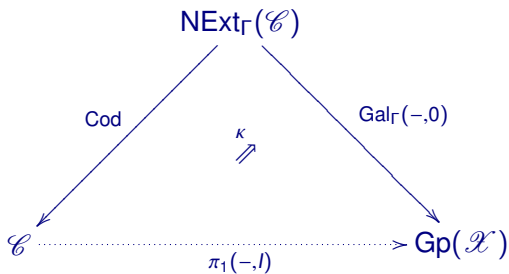
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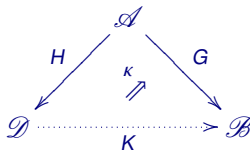
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Kan extension

So indeed we have a Kan extension

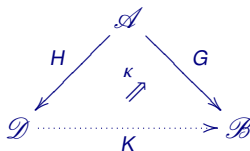


What we used



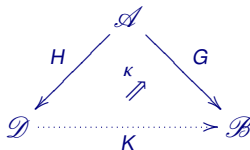
- $H(f) = H(g) \Rightarrow G(f) = G(g)$
- for all $D \in \mathcal{D}$ there is $U \in \mathcal{A}$ with $H(U) = D$ and for all $A \in \mathcal{A}$, $\text{Hom}_{\mathcal{A}}(U, A) \rightarrow \text{Hom}_{\mathcal{D}}(D, HA)$ is surjective.
- Define $K(D) = G(U)$, well-defined and functorial by above properties.
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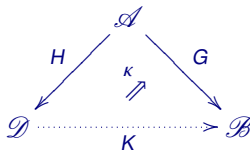
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Next steps

Get also Kan extension

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- For $p: E \rightarrow B$ normal, $U \circ \text{Gal}_\Gamma(p, 0) = \text{Ker } p \cap \text{Ker } \eta_E$.
- So $\iota_p: \text{Ker } p \cap \text{Ker } \eta_E \rightarrow \text{Ker } p$ is a mono.
- We hope that any natural transformation $F \circ \text{Cod} \Rightarrow \text{Ker}$ factors over $U \circ \text{Gal}_\Gamma(-, 0)$. (true in the examples)
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Thanks for listening!

