Groups Example Sheet 3

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1. Consider the Möbius maps $f(z) = e^{2\pi i/n}z$ and $g(z) = 1/z$. Show that the subgroup $\langle f, g\rangle$ of the Möbius group $\mathcal{M}$ is a dihedral group of order $2n$.

2. Let G be a group. If H is a normal subgroup of G and K is a normal subgroup of H, must K be a normal subgroup of G?

3. Show that any group of order 10 is either cyclic or dihedral.

4. Let D_{2n} be the group of symmetries of a regular n-gon.

   (a) Verify that there are 16 subgroups of D_{12}. Which of them are normal?

   (b) For each proper normal subgroup N of D_{12}, identify the quotient D_{12}/N. (Identify means: what standard group is it isomorphic to?)

   (c) More generally, show that **any** subgroup K of rotations is normal in D_{2n}, and identify the quotient D_{2n}/K.

5. Let G be the set of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

   with $x, y, z \in \mathbb{R}$. Show that G is a subgroup of the group of invertible real matrices under multiplication. Let H be the subset of G given by those matrices with $x = z = 0$. Show that H is a normal subgroup of G and identify G/H. (This G is called the *Heisenberg group*.)

6. Take the Heisenberg group as above, but this time with entries in $\mathbb{Z}_3$. Show that every non-identity element of this group has order 3, but the group is not isomorphic to $C_3 \times C_3 \times C_3$.

7. Let K be a normal subgroup of index m in the group G. Show that $a^m \in K$ for any $a \in G$.

8. Let Q be a plane quadrilateral. Show that its group $G(Q)$ of symmetries has order at most 8. For each n in the set $\{1, 2, \ldots, 8\}$, either give an example of a quadrilateral Q with $G(Q)$ of order n, or show that no such quadrilateral can exist.

9. Let G be a finite group and let X be the set of all subgroups of G. Show that G acts on X by $g\colon H \longmapsto gHg^{-1}$ for $g \in G$ and $H \in X$, where $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$. Show that the orbit containing H in this action of G has size at most $|G|/|H|$. If H is a proper subgroup of G, show that there exists an element of G which is contained in no conjugate gHg^{-1} of H in G.

10. Show that D_{2n} has two conjugacy classes of reflections if n is even, but only one if n is odd.

11. Let K be a normal subgroup of order 2 in the group G. Show that K lies in the centre of G; that is, show $kg = gk$ for all $k \in K$ and $g \in G$.

12. Show that the centre of the general linear group $\mathrm{GL}_2(\mathbb{C})$ consists of all non-zero scalar matrices. Identify the centre of the special linear group $\mathrm{SL}_2(\mathbb{C})$.

13. Let p be a prime throughout. Let G be a group of prime power order p^a, with $a > 0$. By considering the conjugation action of G, show that the centre of G is non-trivial.
Show that any group of order p^2 is abelian, and that there are up to isomorphism just two groups of that order for each prime p.

The starred and exploration questions are not necessarily harder, but not necessary for a good understanding of the course. They should only be attempted once you have a solid understanding of the core material. They should also not be attempted to the detriment of later example sheets, or other courses. Exploration questions are meant to lead you as far as you are interested: just start and see how far you can get. There is not necessarily a "full solution".

14. * (Follow-up from Sheet 2 Q14). Exhibit a non-cyclic group of order 55. Prove that any group of order 65 must be cyclic.

15. *

    (a) Let the group G act on the set X. For $g \in G$, define $\text{fix}(g) = \{x \in X \mid g(x) = x\}$. By counting the set $\{(g, x) \in G \times X \mid g(x) = x\}$ in two ways, show that the average fix size $\frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|$ equals the number of orbits of the action.
    (This result is called *Burnside's Lemma*, or *Cauchy-Frobenius Lemma*, or just the *Counting Lemma*.)
    Deduce that if G acts transitively and $|X| > 1$ then there must exist some $g \in G$ with no fixed point.

    (b) In how many distinct ways can the faces of a cube be coloured using at most three colours? What about a dodecahedron? (We regard as equivalent two colourings that can obtained from each other by a rotation.)

16. * Let $\text{SL}_2(\mathbb{R})$ act on $\mathbb{C}_\infty$ via Möbius maps. (Note: this means the entries of the matrix/coefficients in the Möbius map are *real* numbers in this case.) Find the orbit and stabiliser of i and ∞. By considering the orbit of i under the action of the stabiliser of ∞, show that every $g \in \text{SL}_2(\mathbb{R})$ may be written as $g = hk$ with h upper-triangular and $k \in \text{SO}_2$.

17. * *Exploration question* Recall from Sheet 2 Q15, or look up now, the different frieze group examples from the typeset lecture notes, Chapter 4. Picking any of the examples of the later frieze groups with not all symmetries, show that the corresponding subgroup of F_{H} is normal, and determine the quotient. For example, for $F_{\mathsf{V\Lambda}}$, F_{M} and F_{Z}, which are all isomorphic to D_∞, do you get the same quotients or different quotients? Can you find an analogue with subgroups of $\mathbb{Z}$ which are isomorphic as groups but result in different quotients of $\mathbb{Z}$? Can you also find some analogues considering subgroups of D_{2n}?