Higher extensions and the relative Kan property

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joint work with Tomas Everaert and Tim Van der Linden

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Outline

1. Higher extensions
   - Definitions
   - The first three axioms
   - Symmetry
2. $\mathcal{E}$-resolutions
   - Definition
   - Truncations
   - Resolutions and extensions
3. The relative Kan property
   - Relative Mal'tsev Categories
   - The relative Kan property
   - Adding split epis

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Higher arrows

Formal definition:

$$\begin{aligned} \mathsf{Arr}^0\mathcal{A} &= \mathcal{A}, \\ \mathsf{Arr}\mathcal{A} &= \mathsf{Fun}(2^{\mathrm{op}}, \mathcal{A}) \\ \mathsf{Arr}^{n+1}\mathcal{A} &= \mathsf{Arr}\mathsf{Arr}^n\mathcal{A} \end{aligned}$$

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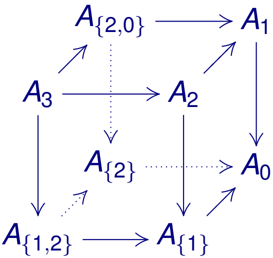

Double extensions

$\mathcal{E}$ a class of extensions. Double extension:

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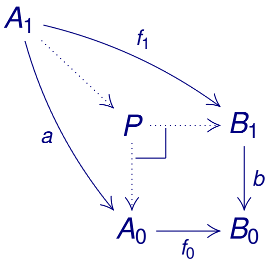

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Higher extensions

- $\mathcal{E}^1$ class of double extensions.
- Inductively get $\mathcal{E}^n = (\mathcal{E}^{n-1})^1$, class of n-fold extensions.
- $\mathrm{Ext}\mathcal{A}$ full subcat of $\mathrm{Arr}\mathcal{A}$ determined by $\mathcal{E}$,
- similarly $\mathrm{Ext}^n\mathcal{A}$ determined by $\mathcal{E}^{n-1}$.

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- $(\mathcal{A}, \mathcal{E})$
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The first three axioms

Let $(\mathcal{A}, \mathcal{E})$ satisfy

(E1) $\mathcal{E}$ contains all isomorphisms;
(E2) pullbacks of extensions exist in $\mathcal{A}$ and are extensions;
(E3) $\mathcal{E}$ is closed under composition.

If $(\mathcal{A}, \mathcal{E})$ satisfies these, then so does $(\mathrm{Ext}\mathcal{A}, \mathcal{E}^1)$.

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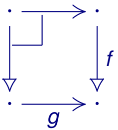

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Examples

- absolute case: regular epis in a regular category
- Projective classes in a finitely complete category
- Topological groups with morphisms which are split as morphisms of topological spaces
- R-modules with morphisms split in Ab
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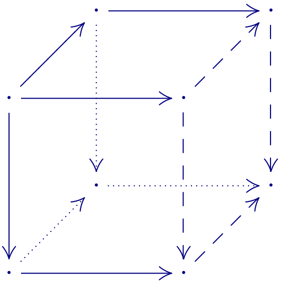
Alternative definition for extensions

Another way of looking at extensions:

- all codomain $(n-1)$-cubes are extensions;
- comparison to limit without "initial corner" is also extension.



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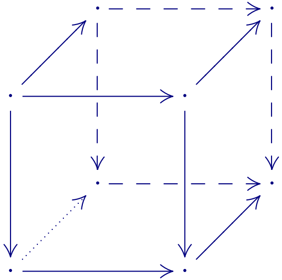

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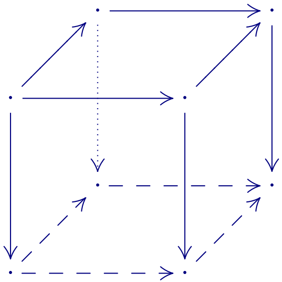

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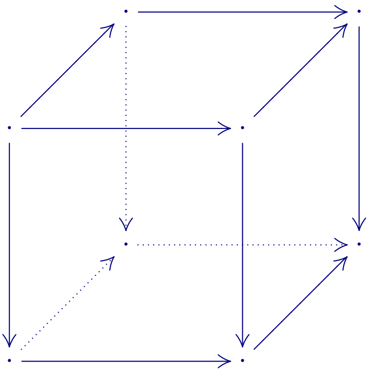

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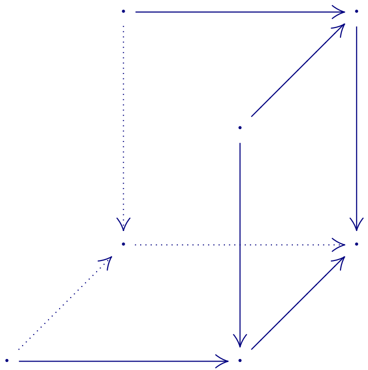

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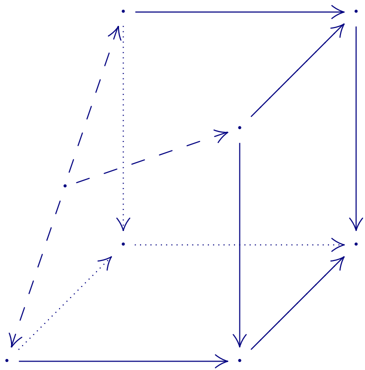

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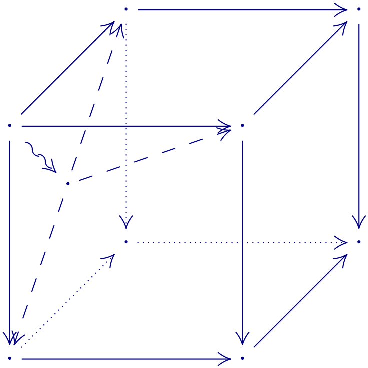

Extensions are symmetric

This makes it easy to see the symmetry of higher extensions:

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ {\scriptstyle a}\downarrow & \Rightarrow & \downarrow{\scriptstyle b} \\ A_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

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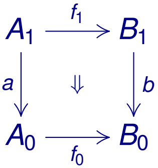

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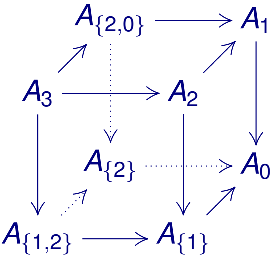

Outline

$\mathcal{E}$-semi-simplicial objects

A (semi)-simplicial object $\mathbb{A}$ is an $\mathcal{E}$-(semi)-simplicial object when all face maps ∂_i are in $\mathcal{E}$.

$$\cdots A_2 \underset{\partial_2}{\overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}\!\!\!\rightarrow}} A_1 \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} A_0$$

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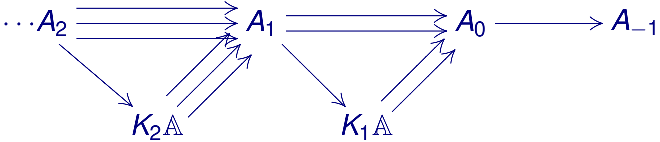
$\mathcal{E}$-resolutions

Factor $\mathcal{E}$-(semi)-simplicial object over its simplicial kernels:



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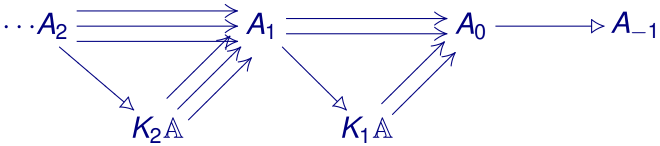

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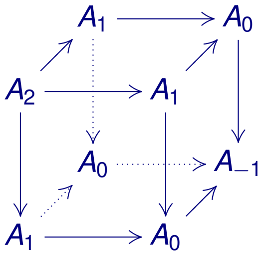

Resolutions and extensions

Theorem

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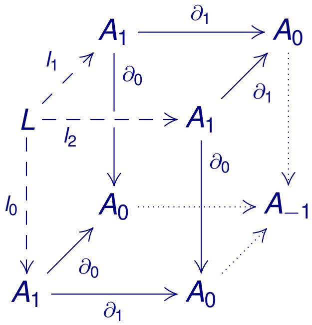

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The relative Mal'tsev axiom

We now add axioms

(E4) if $f \in \mathcal{E}$ and $g \circ f \in \mathcal{E}$ then $g \in \mathcal{E}$; $\quad \cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$

(E5) the $\mathcal{E}$-Mal'tsev condition:

Given a split epi of extensions in $\mathcal{A}$ with a and b also extensions, the square is a double extension.

(F) if f factors as $f = em$ with m mono and $e \in \mathcal{E}$, then also as

$f = m'e'$ with m' mono, $e' \in \mathcal{E}$

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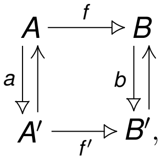

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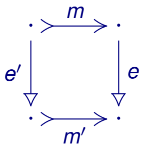

Relative Mal'tsev category

A relative Mal'tsev category is a pair $(\mathcal{A}, \mathcal{E})$, where $\mathcal{A}$ is a category with finite products and $\mathcal{E}$ a class of regular epimorphisms in $\mathcal{A}$, which satisfies (E1)–(E5) and (F).

Axiom (E5)

Under (E1)–(E4), the axiom (E5) implies

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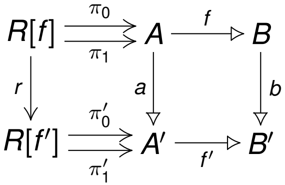

with $a,\ b,\ f,\ f' \in \mathcal{E}$. Then $r \in \mathcal{E} \Leftrightarrow (f, f') \in \mathcal{E}^1$;

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Going up

If $(\mathcal{A}, \mathcal{E})$ satisfies (E1)–(E5), so does $(\mathrm{Ext}\mathcal{A}, \mathcal{E}^1)$.

But (F) does not go up in general:

- If $\mathcal{A}$ semi-abelian, $\mathcal{E}$ regular epis, (F) goes up one step.
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- If $\mathcal{A}$ semi-abelian, $\mathcal{E}$ regular epis, (F) goes up one step.
- If $(\mathcal{A}, \mathcal{E})$ as above with non-trivial abelian object, (F) does not go up two steps.
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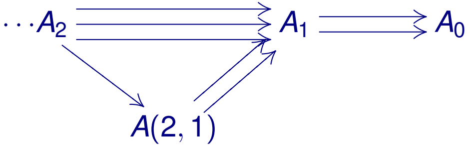
Horn objects

Horn objects $A(n,k)$:
universal object of "collection of horns" in the simplicial object $\mathbb{A}$.



The relative Kan property

- Horn objects exist when $\mathbb{A}$ is an $\mathcal{E}$-semi-simplicial object.
- An $\mathcal{E}$-semi-simplicial object is $\mathcal{E}$-Kan if all comparison maps

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Why "Mal'tsev"

Theorem

When $\mathcal{A}$ has finite products and $\mathcal{E}$ is a class of regular epimorphisms satisfying (E1)–(E4) and (F), then the following are equivalent:

1. *(E5) holds;*
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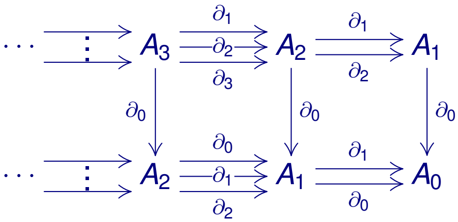
"absolute" result (2)$\Leftrightarrow$(3) by Carboni, Kelly, Pedicchio, 1993

Proof sketches

Proof of (1) to (2) doesn't need (F), uses

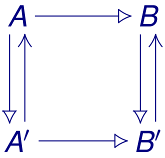


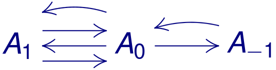
and induction, so needs (E1)–(E5) to go up.

Proof sketches

For (2) to (1) need (F):

- Construct truncated $\mathcal{E}$-simplicial object with contraction;
- this extends to contractible simplicial object which is $\mathcal{E}$-Kan;
- $\mathcal{E}$-Kan + contractible $\Rightarrow$ $\mathcal{E}$-resolution (uses (F)).





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For (2) $\Leftrightarrow$ (3) use (almost) same proof as in "absolute" case, using e.g.

- Every reflexive $\mathcal{E}$-relation is an equivalence $\mathcal{E}$-relation iff $RS = SR$ for any equivalence $\mathcal{E}$-relations R and S.

An $\mathcal{E}$-relation is a relation (R, r_0, r_1) with r_0 and r_1 in $\mathcal{E}$.

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Examples

- absolute case: regular epis in a regular Mal'tsev category
- relative homological and semi-abelian categories (T. Janelidze)
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When all split epis are in $\mathcal{E}$, get

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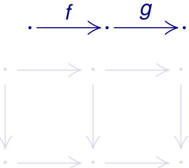

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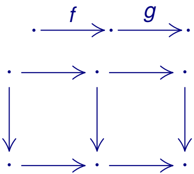

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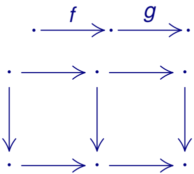

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Given (E1)–(E3)

- Being an extension is symmetric.
- $\mathbb{A}$ is a resolution if and only if all truncations are extensions.

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- $\mathbb{A}$ is an $\mathcal{E}$-resolution if and only if its Moore complex is $\mathcal{E}$-exact;
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Thank you for listening!