Universal Properties

A categorical look at undergraduate algebra and topology

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1. Category Theory
   - Maths is Abstraction
   - Category Theory: more abstraction
2. Universal Properties
   - Within one category
   - Mixing categories

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   - Within one category
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What is Abstraction?

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- Take example/situation/idea.
- Determine some (important) properties.
- “Lift” those away from the example/situation/idea.
- Work with abstracted properties.
- Should get many more examples which also fit these “lifted” properties.

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Examples

- My pet and my friend's pet are both cats.
- Cats, dogs, dolphins are all mamals.
- My home, my old school, the maths department are all buildings.

Numbers

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- “3 apples” $\longrightarrow$ “3”

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After that also (not necessarily in this order)

- negative numbers (abstraction of debt?)
- rational numbers (abstraction of proportions)
- real numbers (abstraction of lengths)

More examples

Groups

- Addition in $\mathbb{Z}$, "clock" addition $\pmod{n}$ and composing symmetries have similar properties.
- Isolate the properties.
- Define an abstract group.
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Equivalence relations

- Study equality, congruence $\pmod{n}$ and "having same image under a function".
- Isolate: reflexivity, symmetry, transitivity.
- Define equivalence relation.
- Work with the abstract idea rather than one example .....

One more level of abstraction

We notice throughout our studies that certain objects come with special maps:

objects	**“structure preserving” maps**
sets	functions
groups	group homomorphisms
rings	ring homomorphisms
modules/vector spaces	linear maps
topological spaces	continuous maps

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- There is an identity:

$$A \xrightarrow{1_A} A \xrightarrow{f} B = A \xrightarrow{f} B = A \xrightarrow{f} B \xrightarrow{1_B} B$$

- Composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

Definition of a category

A category $\mathcal{C}$ consists of

- a collection $\mathrm{ob}\mathcal{C}$ of objects $A, B, C, \ldots$ and
- for each pair of objects $A, B \in \mathrm{ob}\mathcal{C}$, a collection $\mathcal{C}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$ of morphisms $f \colon A \longrightarrow B$,

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- for each $A \in \mathrm{ob}\mathcal{C}$, a morphism $1_A \colon A \longrightarrow A$, the identity,
- for each tripel $A, B, C \in \mathrm{ob}\mathcal{C}$, a composition

$$\circ \colon \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \longrightarrow \mathrm{Hom}(A, C)$$
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such that the following axioms hold:

1. Identity: For $f\colon A \longrightarrow B$ we have $f \circ 1_A = f = 1_B \circ f$.
2. Associativity: For $f\colon A \longrightarrow B$, $g\colon B \longrightarrow C$ and $h\colon C \longrightarrow D$ we have $h \circ (g \circ f) = (h \circ g) \circ f$.

What is Category Theory?

- One more level of abstraction.
  Category Theory is "mathematics about mathematics".

addition, symmetries → groups

equality and congruence → equivalence relations

whole numbers → ring theory

ring theory ↓ categories

- A language for mathematicians.
- A way of thinking.

Categorical point of view

In category theory:

We are not only interested in objects (such as sets, groups, ...), but how different objects of the same kind *relate* to each other. We are interested in global structures and connections.

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Motto of category theory

We want to really understand how and why things work, so that we can present them in a way which makes everything “look obvious”.

Examples of categories

- Any collection of sets with a certain structure and structure-preserving maps will form a category.

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But also:

- A group G is a one-object category with the group elements as morphisms:
  - $e \in G$ is identity morphism.
  - group multiplication is composition.
- A poset P is a category:
  - The elements of P are the objects.
  - $\mathrm{Hom}(x, y)$ has one element if $x \leq y$, empty otherwise.
  - Reflexivity gives identities.
  - Transitivity gives composition.

Universal Property Template

Template

$\mathcal{P}$ some property. A particular X is universal for $\mathcal{P}$ if it has the property $\mathcal{P}$, and if any Y also has property $\mathcal{P}$, then there is a unique map between X and Y which “fits with the property $\mathcal{P}$”.

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Note: could be unique map $X \longrightarrow Y$ or $Y \longrightarrow X$. We specify this for each particular case.

Terminal objects

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Examples

- Sets X: exactly one function $X \longrightarrow \{*\}$.
- Groups G: exactly one group hom $G \longrightarrow 0 = \{e\}$.
- Vector spaces V: exactly one linear map $V \longrightarrow 0$.
- Top. spaces X: exactly one continuous map $X \longrightarrow \{*\}$.

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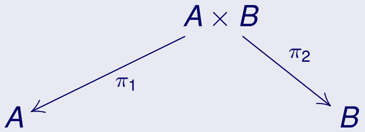
- Rings R: exactly one ring homomorphism $\mathbb{Z} \longrightarrow R$.
- Sets X: exactly one function $\varnothing \longrightarrow X$.
- Topological spaces: also $\varnothing$.

Products

Universal property of a product



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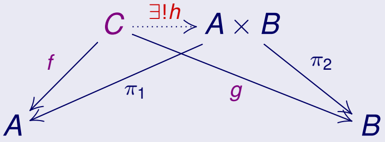

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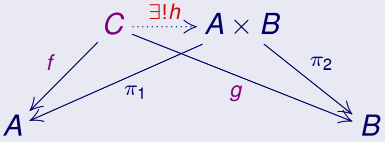

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- Groups: cartesian product with pointwise group structure.

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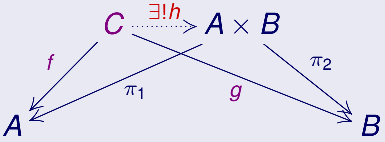

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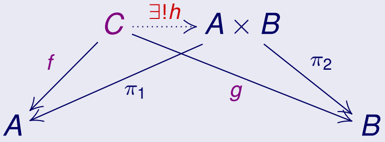

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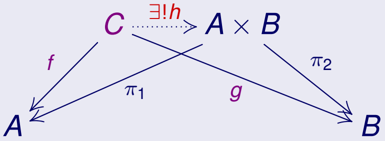

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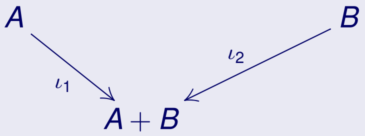
Coproducts

Universal property of a coproduct



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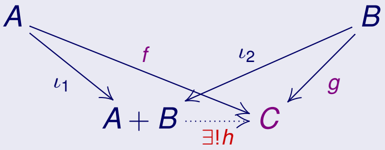

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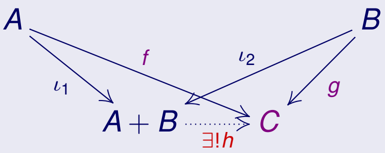

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Examples

- disjoint union of sets $A \coprod B$.
- disjoint union of topological spaces.
- free product of groups $G * H$.
- (external) direct sum of modules $M \oplus N = M \times N$.

A stranger example

Poset as category: $\mathrm{Hom}(x, y)$ has one element if $x \leq y$, empty otherwise.

Universal properties in a poset

- Terminal object is “top element” (if it exists).
- Initial object is “bottom element” (if it exists).
- Products are meets (e.g. in a powerset: intersection).
- Coproducts are joins (e.g. in a powerset: union).

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- Similarly $1_Y = f \circ g$.
- So $X \cong Y$.

Turning around arrows

Initial is "opposite" of terminal

- Terminal T: for all A, $\exists!$ map $A \longrightarrow T$.
- Initial I: for all A, $\exists!$ map $A \longleftarrow I$.

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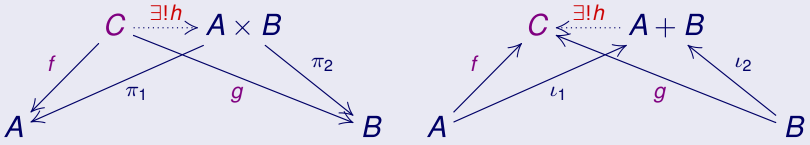
Coproduct is “opposite” of product



Coinciding properties

Zero objects

- For groups and modules, initial = terminal.
- Define zero-object 0 to be both initial and terminal.
- Gives at least one map between any two objects:

$$A \longrightarrow 0 \longrightarrow B$$

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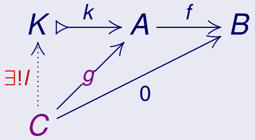
Direct products

- Direct product is both product and coproduct.
- E.g. direct sum of modules (vector spaces, abelian groups...)

Kernels

Universal property of a kernel



Kernel of f is universal map whose post-composition with f is zero.

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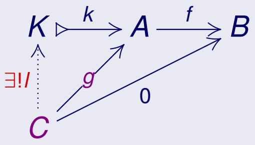

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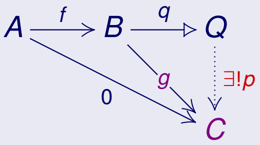
In terms of elements

$K = \{k \in A \mid f(k) = 0\}$, k the inclusion into A.

Cokernels: “turn around the arrows”

Universal property of a cokernel



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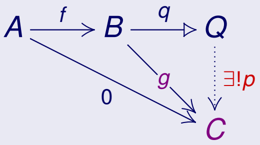

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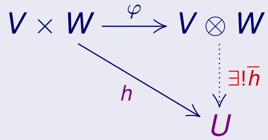
In modules/vector spaces/abelian groups

$Q = B/\mathrm{Im}(f) = \{b + \mathrm{Im}(f)\}$, q the quotient map.

$$A \twoheadrightarrow \mathrm{Im}(f) \rightarrowtail B \twoheadrightarrow B/\mathrm{Im}(f)$$

Tensor Product

Tensor Product of Vector Spaces/Modules



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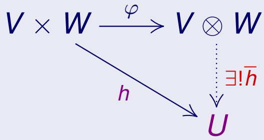

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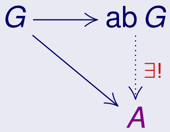
Construction

- Actual construction is complicated and slightly tedious.
- Working with universal property is often easier than with the elements.

Abelianisation

Abelianisation of a group



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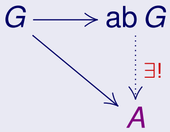

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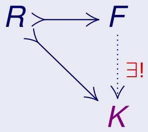
Construction

- $\text{ab}\, G = G/[G, G]$
- $[G, G]$ is commutator: normal subgroup generated by all $aba^{-1}b^{-1}$.

Field of fractions

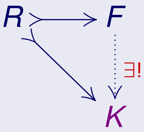
Field of fractions of an integral domain



Every injective ring hom to a field K factors uniquely through the field of fractions.
"Smallest field into which R can be embedded."

Field of fractions

Field of fractions of an integral domain



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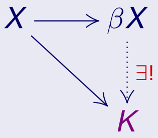
Construction

- $F = \{(a, b) \in R \times R \mid b \neq 0\}/\sim$
- equivalence relation $\sim$ is $(a, b) \sim (c, d)$ iff $ad = bc$.

Stone-Čech Compactification

Compactification of a topological space



Every continuous map to a compact Hausdorff space K factors uniquely through the Stone-Čech compactification.

Stone-Čech Compactification

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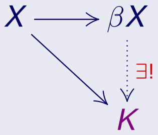

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Generalisation

Abelianisation and Stone-Čech compactification are examples of adjunctions: very important concept in Category Theory.

Why bother?

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- Useful: e.g. to show two objects are isomorphic, show they satisfy same universal property.

Thanks for listening!