# The commutator condition for higher central extensions

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joint work with Diana Rodelo

Fonds de la Recherche Scientifique–FNRS
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- aim of the talk: explaining some (co)homological consequences of the *Smith is Huq* property
- main result based on unpublished work of Tomas Everaert's
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# Overview

# First problem: homology

Theorem [Everaert, Gran & VdL, 2008]

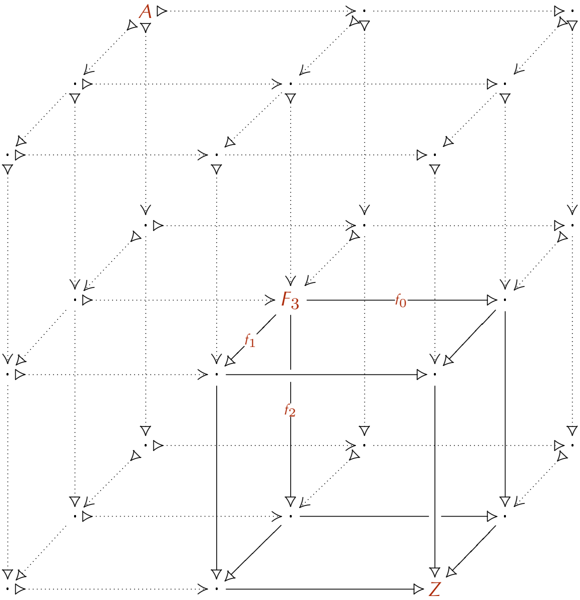
In a semi-abelian monadic category $\mathcal{A}$, for any $n$-presentation $F$ of $Z$,

$$\mathrm{H}_{n+1}(Z, \mathbf{Ab}(\mathcal{A})) \cong \frac{[F_n, F_n] \wedge \bigwedge_{i \in n} \mathrm{Ker}(f_i)}{L_n[F]}.$$

Here $F_n$ is the initial object of $F$ and the $f_i$ are the initial arrows. $\square$

$A = \bigwedge_{i \in n} \mathrm{Ker}(f_i)$ is called the **direction** of $F$.

# A three-fold (central) extension of $Z$ by $A$



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The denominator is not explicit!

*$L_n[F]$ is the smallest normal subobject of $F_n$ which, when divided out, makes $F$ central.*

In the examples it is a join of commutators.

Solution

Characterise higher central extensions in terms of commutators.

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# The commutator condition (CC)

## Definition

An $n$-fold extension $F$ is **H-central** when

$$\Big[\bigwedge_{i\in I} \mathrm{Ker}(f_i), \bigwedge_{i\in n\setminus I} \mathrm{Ker}(f_i)\Big] = 0$$

for all $I \subseteq n$.

↖ **Huq or Higgins commutators**

## Definition

A semi-abelian category satisfies the **commutator condition (CC)** when H-centrality is equivalent to centrality.

Degree-wise:
a semi-abelian category satisfies **(CC$n$)** when
an $n$-fold extension is H-central iff it is central.

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# Second problem: cohomology

Proposition [Rodelo & VdL, 2011]

Let $F$ be an $n$-fold extension. Then $1 \Rightarrow 2 \Rightarrow 3$:

1. $F$ is central; ← **Galois theory**
2. $F$ is an $n$-torsor; ← **Duskin–Glenn cohomology**
3. $F$ is H-central. □

(CC) says $3 \Rightarrow 1$, so:

Theorem [Rodelo & VdL, 2011]

Let $Z$ be an object, $A$ an abelian object in a semi-abelian category $\mathcal{A}$. When $\mathcal{A}$ has (CC), there is an isomorphism

$$\mathrm{H}^{n+1}(Z, A) \cong \mathrm{Centr}^n(Z, A)$$

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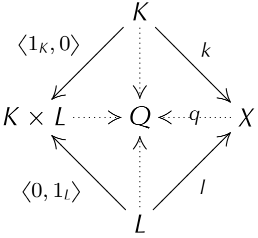


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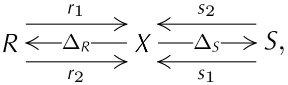


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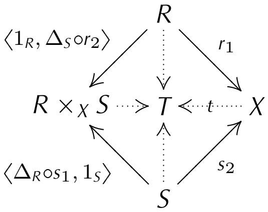

# Two commutators

$$K \overset{r_1 \circ \ker(r_2)}{\rhd\!\!\longrightarrow} X \overset{s_1 \circ \ker(s_2)}{\longleftarrow\!\!\lhd} L \text{ normalisations of } R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X \underset{s_1}{\overset{s_2}{\leftleftarrows}} S$$

- $[R, S]^{\mathrm{S}} = \Delta_X$ implies always $[K, L] = 0$ [Bourn & Gran, 2002].
- The converse is the **Smith is Huq** condition **(SH)**.

# The situation in degree 1

$$X \xrightarrow{f} Z \quad \text{extension}, \quad K = \mathrm{Ker}(f) \quad \text{and} \quad R = \mathrm{R}[f]$$

is H-central when $[K, X] = 0$

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# Main result

### Question

Is (CC) a higher-dimensional version of (SH)?

### Answer

No!

### Theorem

If a semi-abelian category has (CC2) then it has (CC).
In particular, (SH) $\Rightarrow$ (CC). $\square$

- So (CC) stays within bounds:
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# Overview

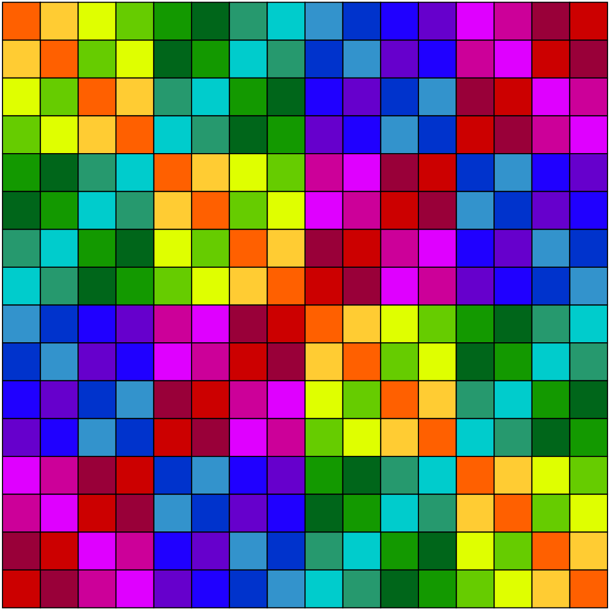
# The category **Loop** of loops and loop homomorphisms

- A **loop** is a quasigroup with a neutral element: an algebraic structure $(X, \cdot, \backslash, /, 1)$ that satisfies $x \cdot 1 = x = 1 \cdot x$ and

$$y = x \cdot (x \backslash y) \qquad y = x \backslash (x \cdot y)$$
$$x = (x/y) \cdot y \qquad x = (x \cdot y)/y$$

- semi-abelian variety: $n = 1, \quad t(x, y) = x \cdot y, \quad t_1(x, y) = x/y$
- associative loop = group
- multiplication table of a loop = Latin square with unit
- associator: $[\![x, y, z]\!] = ((x \cdot y) \cdot z)/(x \cdot (y \cdot z))$ for $x, y, z \in X$

# A counterexample



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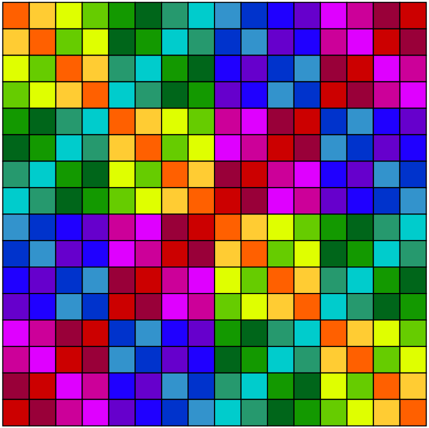
$X$ is the (commutative) loop given by the Latin square



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- indeed $1 \neq [\![k, l, x]\!]$ while $([\![k, l, x]\!], 1) \in [R, S]^{S}$

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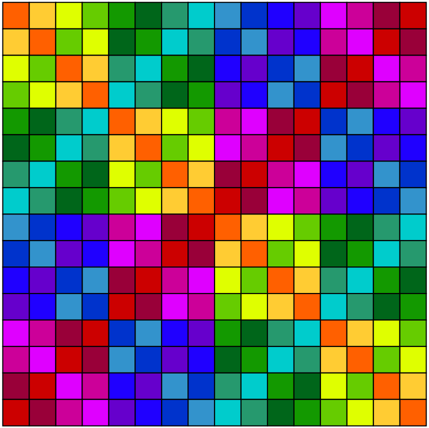

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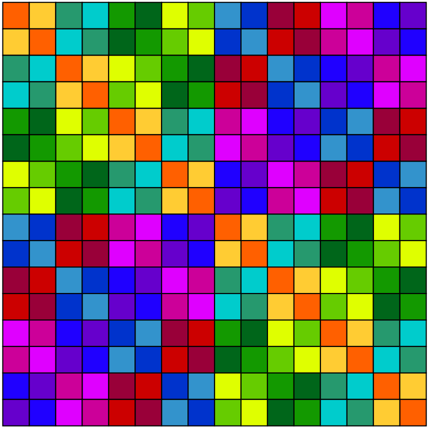

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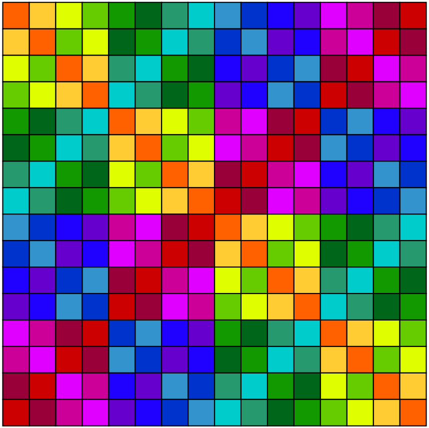

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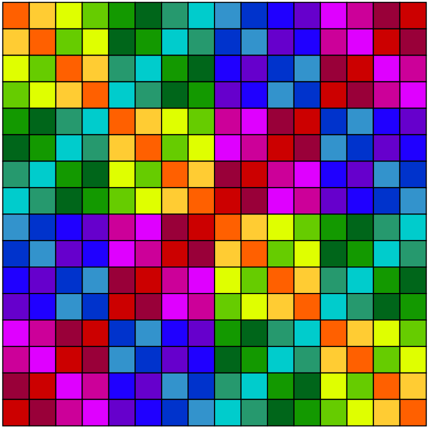

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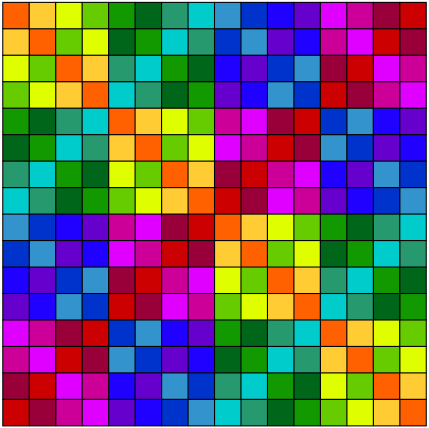

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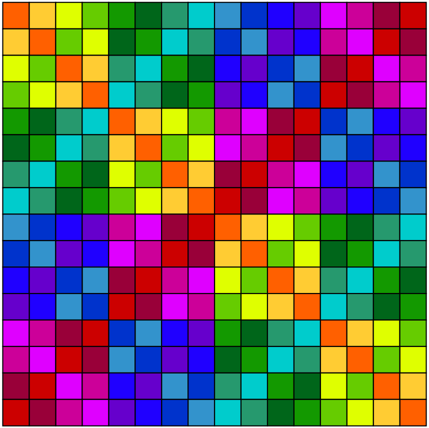

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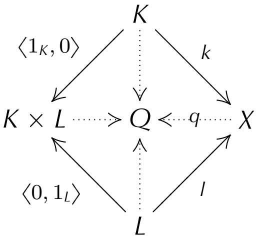


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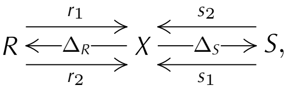


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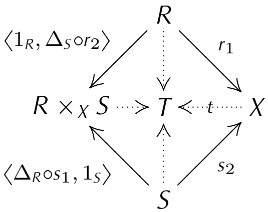

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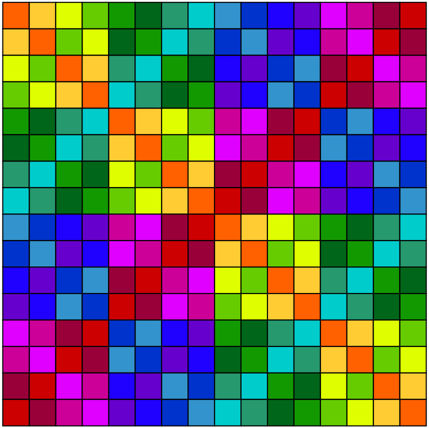

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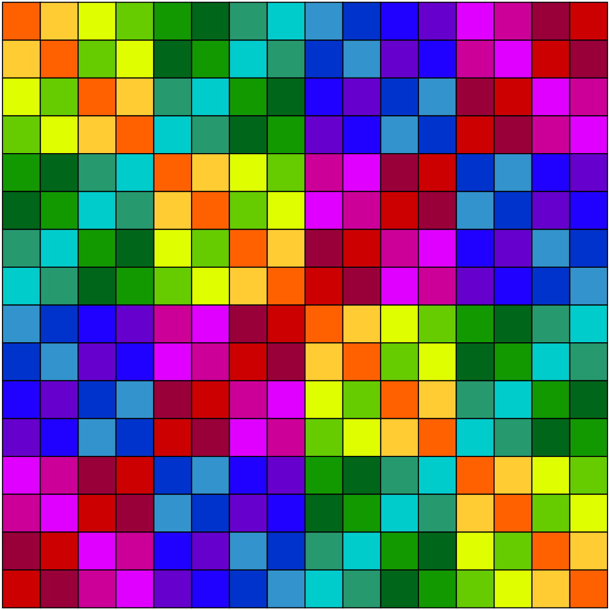
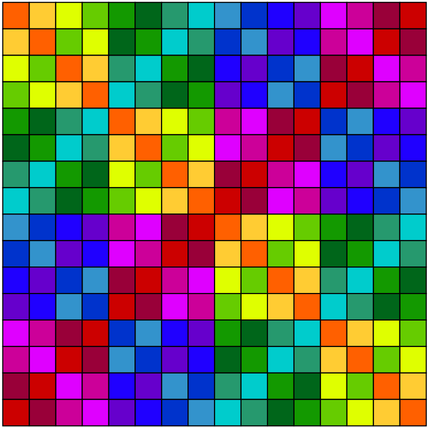
The semi-abelian variety **Loop** does not satisfy (CC).
In fact, also **CLoop** does not satisfy (CC) or (SH)!



# A counterexample

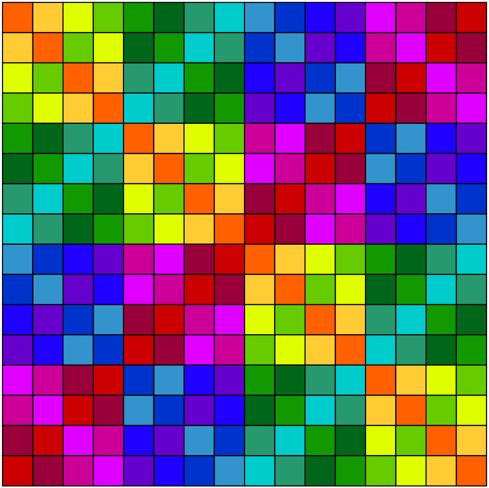
$X$ is the (commutative) loop given by the Latin square



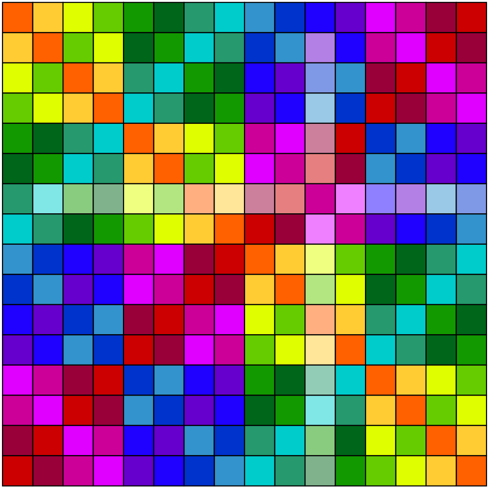
- $K =$  and $L =$  are normal in $X$
- $[K, L] = 0 = [K \wedge L, X]$ but $[R, S]^{\mathrm{S}}$ **is non-trivial**
- indeed $1 \neq [\![k, l, x]\!]$ while $([\![k, l, x]\!], 1) \in [R, S]^{\mathrm{S}}$

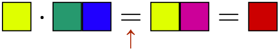
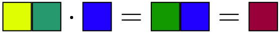
  $1 =$  and $k =$  $l =$  $x =$ 

# A counterexample

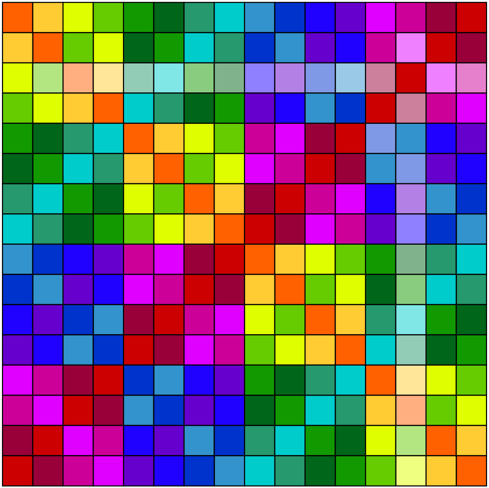


# A counterexample

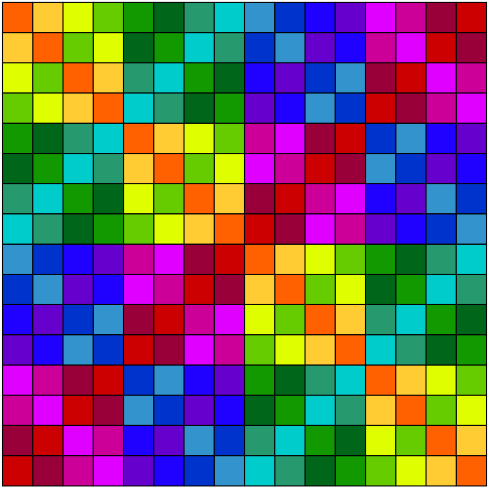


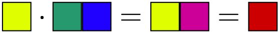
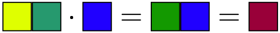
 while 

# A counterexample

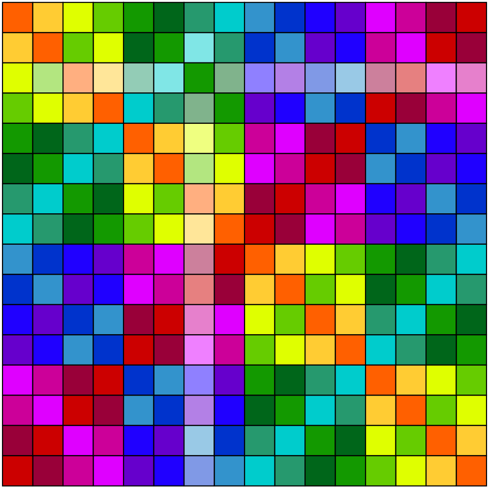


# A counterexample

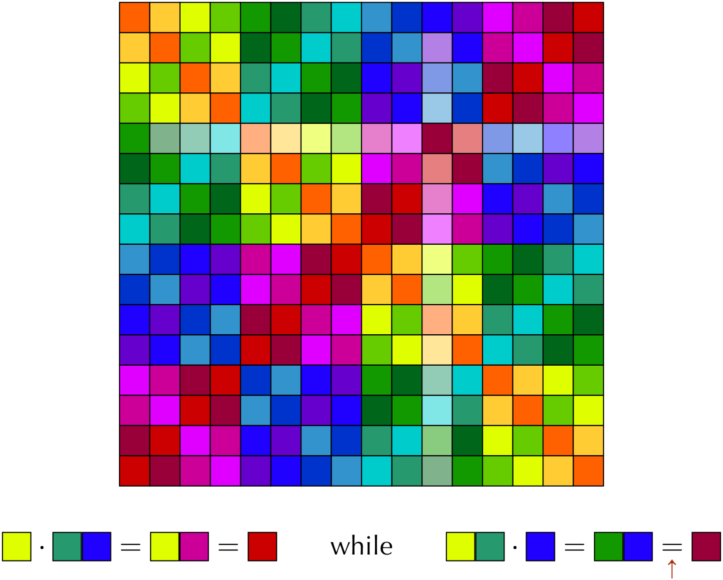


 while 

# A counterexample



# A counterexample



# A counterexample