Multivariable adjunctions and mates

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PSSL93

Adjunctions

Mates

Adjunctions

Mates

Organising framework

Double categories

Adjunctions $\longrightarrow$ Multivariable adjunctions

Mates

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Mates $\longrightarrow$ "Parametrised mates"

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Plan

1. Mates via double categories

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1. Mates via double categories
2. Multivariables

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1. Mates via double categories
2. Multivariables
3. Cyclic double multicategories

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Plan

1. Mates via double categories
2. Multivariables
3. Cyclic double multicategories
4. Application

1. Mates via double categories

Given adjunctions

$$
\begin{tikzcd}
A \arrow[d, shift right=2, "F"'] \arrow[d, phantom, "\dashv"] \\
B \arrow[u, shift right=2, "G"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[d, shift right=2, "F'"'] \arrow[d, phantom, "\dashv"] \\
B' \arrow[u, shift right=2, "G'"']
\end{tikzcd}
$$

Given adjunctions

$$\begin{tikzcd} A \arrow[d, shift right=1ex, "F"'] \arrow[d, phantom, "\dashv"] \\ B \arrow[u, shift right=1ex, "G"'] \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[d, shift right=1ex, "F'"'] \arrow[d, phantom, "\dashv"] \\ B' \arrow[u, shift right=1ex, "G'"'] \end{tikzcd}$$

we have a correspondence between natural transformations

$$\begin{tikzcd} A \arrow[r, "S"] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha\ \Swarrow"] & A' \arrow[d, "F'"] \\ B \arrow[r, "T"'] & B' \end{tikzcd}$$

1. Mates via double categories

Given adjunctions

$$
\begin{tikzcd}
A \arrow[d, "F"', shift right=2] \arrow[d, phantom, "\dashv"] \\
B \arrow[u, "G"', shift right=2]
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[d, "F'"', shift right=2] \arrow[d, phantom, "\dashv"] \\
B' \arrow[u, "G'"', shift right=2]
\end{tikzcd}
$$

we have a correspondence between natural transformations

$$
\begin{tikzcd}
A \arrow[r, "S"] \arrow[d, "F"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & A' \arrow[d, "F'"] \\
B \arrow[r, "T"'] & B'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "S"] \arrow[dr, phantom, "\underset{\beta}{\searrow}"] & A' \\
B \arrow[u, "G"] \arrow[r, "T"'] & B' \arrow[u, "G'"']
\end{tikzcd}.
$$

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$$
\begin{tikzcd}
A \arrow[d, shift right=1ex, "F"'] \arrow[d, phantom, "\dashv"] & A' \arrow[d, shift right=1ex, "F'"'] \arrow[d, phantom, "\dashv"] \\
B \arrow[u, shift right=1ex, "G"'] & B' \arrow[u, shift right=1ex, "G'"']
\end{tikzcd}
$$

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$$
\begin{tikzcd}
B \arrow[r, "G"] \arrow[dr, "1"'] & A \arrow[r, "S"] \arrow[d, "F"'] \arrow[dl, phantom, "\Swarrow\varepsilon"] \arrow[dr, phantom, "\Swarrow\alpha"] & A' \arrow[d, "F'"] \arrow[dr, "1"] & \\
& B \arrow[r, "T"'] & B' \arrow[r, "G'"'] \arrow[ur, phantom, "\Swarrow\eta'"] & A'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "S"] \arrow[dr, phantom, "\beta\Searrow"] & A' \\
B \arrow[u, "G"] \arrow[r, "T"'] & B' \arrow[u, "G'"']
\end{tikzcd}.
$$

Given adjunctions

$$
\begin{tikzcd}
A \arrow[d, shift right=1.5ex, "F"'] \arrow[d, phantom, "\dashv"] \\
B \arrow[u, shift right=1.5ex, "G"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[d, shift right=1.5ex, "F'"'] \arrow[d, phantom, "\dashv"] \\
B' \arrow[u, shift right=1.5ex, "G'"']
\end{tikzcd}
$$

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\begin{tikzcd}
A \arrow[r, "S"] \arrow[d, "F"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & A' \arrow[d, "F'"] \\
B \arrow[r, "T"'] & B'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & A \arrow[r, "S"] \arrow[dr, phantom, "\overset{\beta}{\searrow}"] & A' \arrow[r, "F'"] \arrow[dr, phantom, "\overset{\varepsilon'}{\searrow}"] & B' \\
A \arrow[ur, "1"] \arrow[r, "F"'] \arrow[ur, phantom, bend right, "\overset{\eta}{\searrow}"'] & B \arrow[u, "G"] \arrow[r, "T"'] & B' \arrow[u, "G'"'] \arrow[ur, "1"'] &
\end{tikzcd}
$$

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A \arrow[d, shift right=1.5ex, "F"'] \arrow[d, phantom, "\dashv"] \\
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\end{tikzcd}
$$

we have a correspondence between natural transformations

$$
\begin{tikzcd}
B \arrow[r, "G"] \arrow[dr, "1"'] & A \arrow[r, "S"] \arrow[d, "F"] \arrow[dl, phantom, "\varepsilon\Swarrow"] \arrow[dr, phantom, "\alpha\Swarrow"] & A' \arrow[d, "F'"'] \arrow[dr, "1"] & \\
& B \arrow[r, "T"'] & B' \arrow[r, "G'"'] \arrow[ur, phantom, "\Swarrow\eta'"] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[r, "S"] & A' \arrow[r, "F'"] \arrow[dr, phantom, "\Searrow\varepsilon'"] & B' \\
A \arrow[ur, "1"] \arrow[r, "F"'] \arrow[ur, phantom, "\eta\Searrow"'] & B \arrow[u, "G"] \arrow[r, "T"'] \arrow[ur, phantom, "\beta\Searrow"] & B' \arrow[u, "G'"'] \arrow[ur, "1"'] &
\end{tikzcd}.
$$

- directions are crucial

1. Mates via double categories

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A \arrow[d, shift right=2, "F"'] \arrow[d, phantom, "\dashv"] \\
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$$
\begin{tikzcd}
B \arrow[r, "G"] \arrow[dr, "1"'] \arrow[r, phantom, shift right=4, "\Swarrow\varepsilon"] & A \arrow[r, "S"] \arrow[d, "F"] \arrow[dr, phantom, "\alpha\,\Swarrow"] & A' \arrow[d, "F'"] \arrow[dr, "1"] \arrow[d, phantom, shift left=6, "\Swarrow\eta'"] & \\
 & B \arrow[r, "T"'] & B' \arrow[r, "G'"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[r, "S"] \arrow[dr, phantom, "\beta\,\Searrow"] & A' \arrow[r, "F'"] & B' \\
A \arrow[ur, "1"] \arrow[r, "F"'] \arrow[r, phantom, shift left=4, "\eta\Searrow"] & B \arrow[u, "G"] \arrow[r, "T"'] & B' \arrow[u, "G'"'] \arrow[ur, "1"'] \arrow[u, phantom, shift left=6, "\Searrow\varepsilon'"] &
\end{tikzcd}.
$$

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- bijective correspondence

1. Mates via double categories

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$$\begin{tikzcd} A \arrow[d, shift right=2, "F"'] \arrow[d, phantom, "\dashv"] \\ B \arrow[u, shift right=2, "G"'] \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[d, shift right=2, "F'"'] \arrow[d, phantom, "\dashv"] \\ B' \arrow[u, shift right=2, "G'"'] \end{tikzcd}$$

we have a correspondence between natural transformations

$$\begin{tikzcd} B \arrow[r, "G"] \arrow[dr, "1"'] & A \arrow[r, "S"] \arrow[d, "F"'] \arrow[l, phantom, "\Swarrow\varepsilon" near end] \arrow[dr, phantom, "\overset{\alpha}{\Swarrow}"] & A' \arrow[d, "F'"] \arrow[dr, "1"] & \\ & B \arrow[r, "T"'] & B' \arrow[r, "G'"'] \arrow[ur, phantom, "\Swarrow\eta'" near start] & A' \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[r, "S"] \arrow[dr, phantom, "\underset{\beta}{\Searrow}"] & A' \arrow[r, "F'"] \arrow[dr, phantom, "\Searrow\varepsilon'" near start] & B' \\ A \arrow[ur, "1"] \arrow[r, "F"'] \arrow[ur, phantom, "\eta\Searrow" near end] & B \arrow[u, "G"] \arrow[r, "T"'] & B'. \arrow[u, "G'"'] \arrow[ur, "1"'] & \end{tikzcd}$$

- directions are crucial
- bijective correspondence
- respects horizontal and vertical composition

1. Mates via double categories

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$$\begin{tikzcd} A \arrow[d, shift right=2, "F"'] \arrow[d, phantom, "\dashv"] \\ B \arrow[u, shift right=2, "G"'] \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[d, shift right=2, "F'"'] \arrow[d, phantom, "\dashv"] \\ B' \arrow[u, shift right=2, "G'"'] \end{tikzcd}$$

we have a correspondence between natural transformations

$$\begin{tikzcd} B \arrow[r, "G"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=3, "\Swarrow\varepsilon"] & A \arrow[r, "S"] \arrow[d, "F"'] \arrow[dr, phantom, "\Swarrow\alpha"] & A' \arrow[d, "F'"'] \arrow[dr, "1"] \arrow[dr, phantom, shift right=3, "\Swarrow\eta'"] & \\ & B \arrow[r, "T"'] & B' \arrow[r, "G'"'] & A' \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[r, "S"] \arrow[dr, phantom, "\beta\Searrow"] & A' \arrow[r, "F'"] \arrow[dr, phantom, shift left=3, "\Searrow\varepsilon'"] & B' \\ A \arrow[ur, "1"] \arrow[r, "F"'] \arrow[ur, phantom, shift right=3, "\eta\Searrow"] & B \arrow[u, "G"] \arrow[r, "T"'] & B'. \arrow[u, "G'"'] \arrow[ur, "1"'] & \end{tikzcd}$$

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This is neatly summed up using double categories.

Recall: a double category is an internal category in **Cat**.

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- 0-cells A

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- 0-cells
- horizontal and vertical 1-cells

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Recall: a double category is an internal category in **Cat**.

- 0-cells
- horizontal and vertical 1-cells
- 2-cells

$$\begin{tikzcd} A \arrow[r, "S"] \arrow[d, "F"'] \arrow[dr, phantom, "\Downarrow^{\alpha}"] & A' \arrow[d, "F'"] \\ B \arrow[r, "T"'] & B' \end{tikzcd}$$

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Horizontal and vertical composition:

$$\begin{tikzcd} A_1 \arrow[r, "S_1"] \arrow[d, "F_1"'] \arrow[dr, phantom, "\Downarrow\alpha_1"] & A_2 \arrow[r, "S_2"] \arrow[d, "F_2"] \arrow[dr, phantom, "\Downarrow\alpha_2"] & A_3 \arrow[d, "F_3"] \\ B_1 \arrow[r, "T_1"'] & B_2 \arrow[r, "T_2"'] & B_3 \end{tikzcd}
\qquad
\begin{tikzcd} A_1 \arrow[r, "S"] \arrow[d, "F_1"'] \arrow[dr, phantom, "\Downarrow\alpha_1"] & A_2 \arrow[d, "F_2"] \\ B_1 \arrow[r, "T"'] \arrow[d, "H_1"'] \arrow[dr, phantom, "\Downarrow\alpha_2"] & B_2 \arrow[d, "H_2"] \\ C_1 \arrow[r, "U"'] & C_2 \end{tikzcd}$$

1. Mates via double categories

We define double categories $\mathbb{L}\mathbf{Adj}$ and $\mathbb{R}\mathbf{Adj}$

- 0-cells: categories
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- A 2-cell $\begin{tikzcd} A_1 \arrow[r, "S"] \arrow[d, "F_1"', shift right] \arrow[d, phantom, "\dashv"] \arrow[dr, phantom, "\Downarrow"] & A_2 \arrow[d, "F_2"', shift right] \arrow[d, phantom, "\dashv"] \\ B_1 \arrow[u, "G_1"', shift right] \arrow[r, "T"'] & B_2 \arrow[u, "G_2"', shift right] \end{tikzcd}$ is a natural transformation

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  in $\mathbb{L}\mathbf{Adj}$ $\begin{tikzcd} A_1 \arrow[r, "S"] \arrow[d, "F_1"'] \arrow[dr, phantom, "\overset{\alpha}{\swArrow}"] & A_2 \arrow[d, "F_2"] \\ B_1 \arrow[r, "T"'] & B_2 \end{tikzcd}$

1. Mates via double categories

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- a vertical 1-cell $\begin{tikzcd} A \arrow[d, "F"'] \\ B \end{tikzcd}$ is an adjunction $\begin{tikzcd} A \arrow[d, shift right=1.5, "F"'] \arrow[d, phantom, "\dashv"] \\ B \arrow[u, shift right=1.5, "G"'] \end{tikzcd}$
- A 2-cell $$\begin{tikzcd} A_1 \arrow[r, "S"] \arrow[d, shift right=1.5, "F_1"'] \arrow[d, phantom, "\dashv"] \arrow[dr, phantom, "\Searrow"] & A_2 \arrow[d, shift right=1.5, "F_2"'] \arrow[d, phantom, "\dashv"] \\ B_1 \arrow[r, "T"'] \arrow[u, shift right=1.5, "G_1"'] & B_2 \arrow[u, shift right=1.5, "G_2"'] \end{tikzcd}$$ is a natural transformation

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$$\begin{tikzcd} A_1 \arrow[r, "S"] \arrow[d, "F_1"'] \arrow[dr, phantom, "\overset{\alpha}{\Swarrow}"] & A_2 \arrow[d, "F_2"] \\ B_1 \arrow[r, "T"'] & B_2 \end{tikzcd}$$

in $\mathbb{R}\mathbf{Adj}$
$$\begin{tikzcd} A_1 \arrow[r, "S"] \arrow[dr, phantom, "\overset{\alpha}{\Searrow}"] & A_2 \\ B_1 \arrow[r, "T"'] \arrow[u, "G_1"] & B_2 \arrow[u, "G_2"'] \end{tikzcd}$$

Theorem *Kelly–Street*

There is an isomorphism of double categories

$$\mathbb{L}\mathbf{Adj} \xrightarrow{\cong} \mathbb{R}\mathbf{Adj}.$$

- on 0- and 1-cells: identity
- on 2-cells: take mates

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$$\begin{tikzcd} A \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & B \arrow[l, shift left, "G"] \end{tikzcd} \quad \equiv$$

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$$\begin{tikzcd} A_1 \arrow[r, "S"] \arrow[d, "F_1"'] \arrow[dr, phantom, "\Swarrow\alpha"] & A_2 \arrow[d, "F_2"] \\ B_1 \arrow[r, "T"'] & B_2 \end{tikzcd}$$

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$$\begin{tikzcd} A_1 \arrow[r, "S"] \arrow[d, "F_1"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & A_2 \arrow[d, "F_2"] \\ B_1 \arrow[r, "T"'] & B_2 \end{tikzcd} \quad \overset{\text{mate}}{\longmapsto} \quad \begin{tikzcd} A_1 \arrow[r, "S"] \arrow[dr, phantom, "\underset{\bar{\alpha}}{\searrow}"] & A_2 \\ B_1 \arrow[r, "T"'] \arrow[u, "G_1"] & B_2 \arrow[u, "G_2"'] \end{tikzcd}$$

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Theorem *Kelly–Street*

There is an isomorphism of double categories

$$\mathbb{L}\mathbf{Adj} \xrightarrow{\simeq} \mathbb{R}\mathbf{Adj}.$$

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A_1 \arrow[r, "S"] \arrow[d, "F_1"'] \arrow[dr, phantom, "\alpha \swArrow"] & A_2 \arrow[d, "F_2"] & & A_1 \arrow[r, "S"] \arrow[dr, phantom, "\bar\alpha \seArrow"] & A_2 & & A_1^\bullet \arrow[r, "S^\bullet"] \arrow[dr, phantom, "\nwArrow \bar\alpha^\bullet"] & A_2^\bullet & & B_1^\bullet \arrow[r, "T^\bullet"] \arrow[d, "G_1^\bullet"'] \arrow[dr, phantom, "\bar\alpha^\bullet \swArrow"] & B_2^\bullet \arrow[d, "G_2^\bullet"] \\
B_1 \arrow[r, "T"'] & B_2 \arrow[r, phantom, "\overset{\text{mate}}{\longmapsto}"] & & B_1 \arrow[r, "T"'] \arrow[u, "G_1"] & B_2 \arrow[u, "G_2"'] \arrow[r, phantom, "\overset{\text{op}}{\longmapsto}"] & & B_1^\bullet \arrow[r, "T^\bullet"'] \arrow[u, "G_1^\bullet"] & B_2^\bullet \arrow[u, "G_2^\bullet"'] \arrow[r, phantom, "\overset{\text{flip}}{\longmapsto}"] & & A_1^\bullet \arrow[r, "S^\bullet"'] & A_2^\bullet
\end{tikzcd}$$

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There is an isomorphism of double categories

$$\mathbb{L}\mathbf{Adj} \xrightarrow{\cong} \mathbb{R}\mathbf{Adj}.$$

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1. Mates via double categories

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—the 1-ary part of a cyclic structure.

2. Multivariables

Key 2-variable example: tensor-hom-cotensor

A monoidal category $\mathcal{V}$ is biclosed if

- $\forall b \in \mathcal{V}$ the functor $\_ \otimes b$ has a right adjoint "hom"
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$$\cong$$

—natural in *all* variables.

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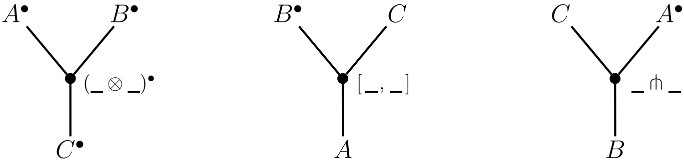

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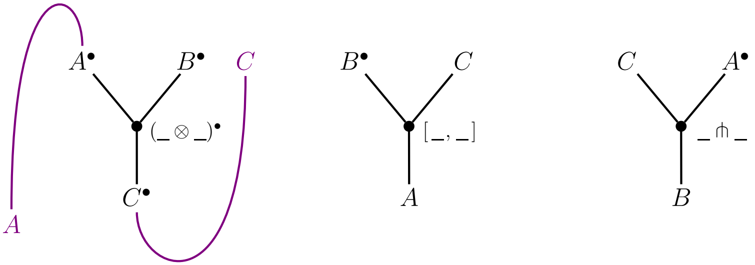

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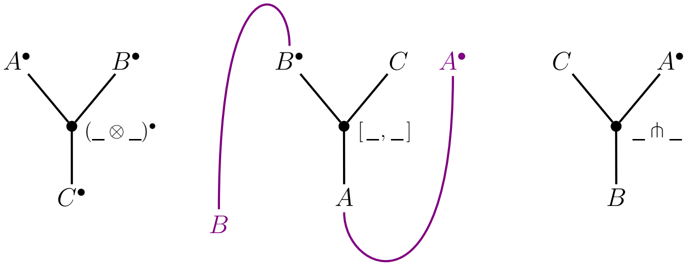

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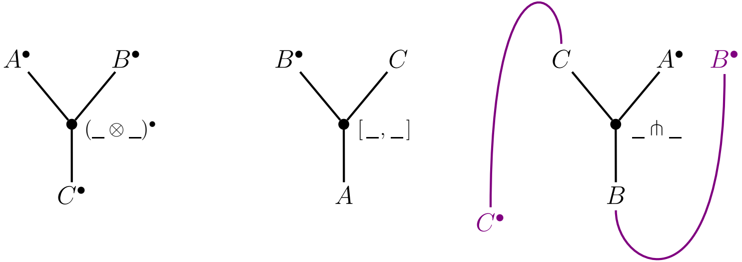

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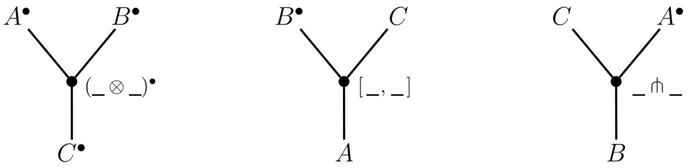

Note: it is enough to specify these to get everything else.

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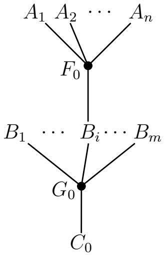
natural in *all* variables and commuting like billy-o.

Results

- we can compose them



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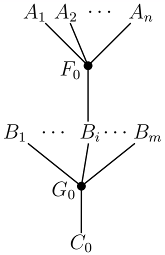

- we get a functorial multivariable mates correspondence

$$\begin{tikzcd}
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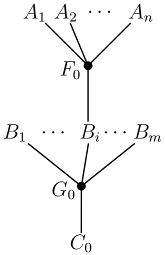

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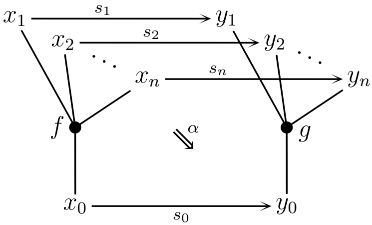

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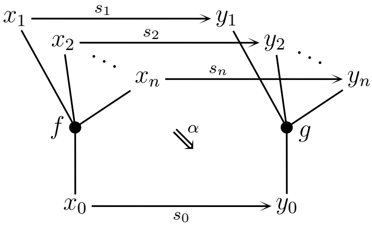

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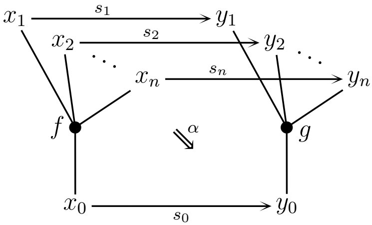

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that is

$$\mathbb{M}\mathbf{Adj}_v(A_1, \ldots, A_n; A_0^\bullet) \cong \mathbb{M}\mathbf{Adj}_v(A_2, \ldots, A_0; A_1^\bullet)$$

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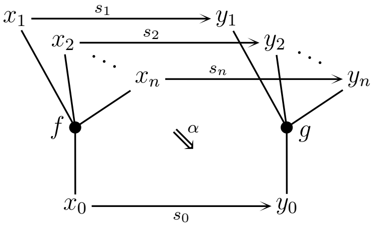

- 0-cells and horizontal 1-cells
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Instead of $\mathbb{L}\mathbf{Adj} \cong \mathbb{R}\mathbf{Adj}$
we want a multivariable version of

$$\mathbb{L}\mathbf{Adj}_v(A, B) \cong \mathbb{L}\mathbf{Adj}_v(B^\bullet, A^\bullet)$$

$$\mathbb{L}\mathbf{Adj}_2(S, T) \cong \mathbb{L}\mathbf{Adj}_2(T^\bullet, S^\bullet)$$

that is

$$\mathbb{M}\mathbf{Adj}_v(A_1, \ldots, A_n; A_0^\bullet) \cong \mathbb{M}\mathbf{Adj}_v(A_2, \ldots, A_0; A_1^\bullet)$$

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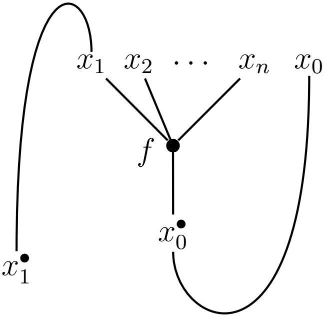
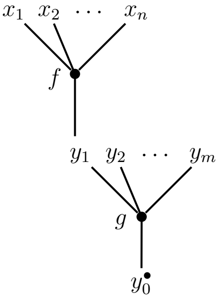
we depict σf as



3. Cyclic double multicategories

satisfying axioms



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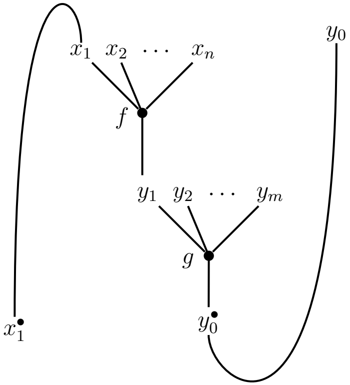

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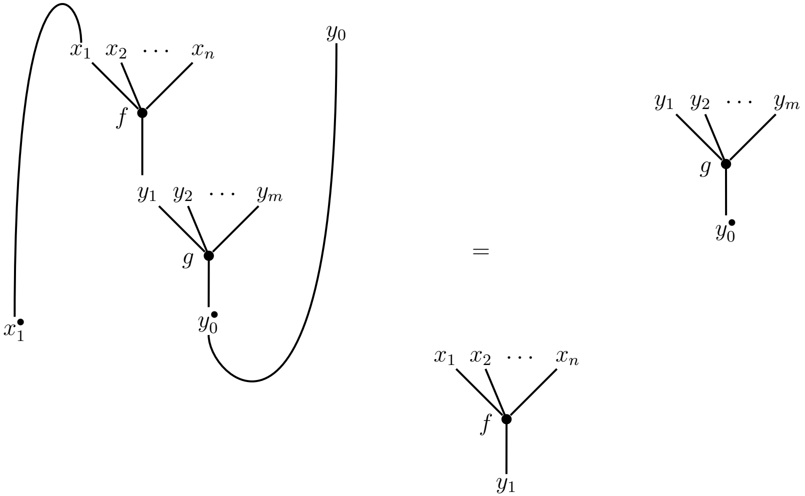

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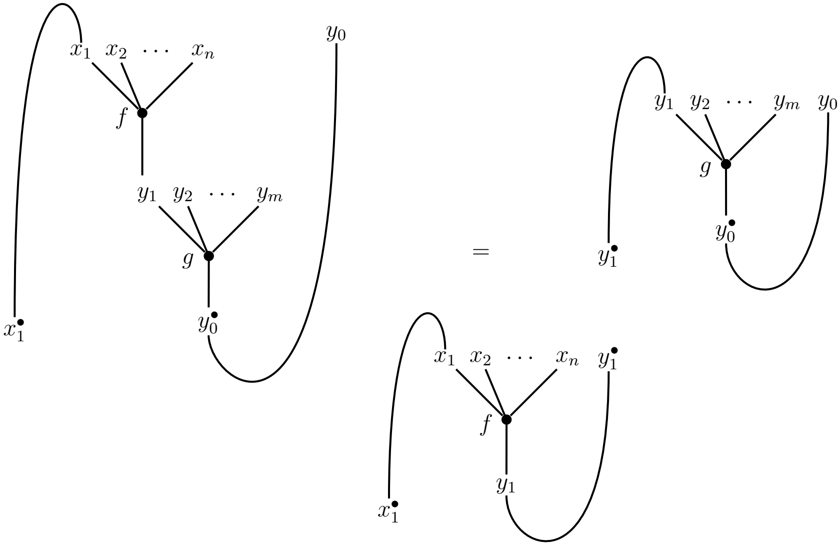

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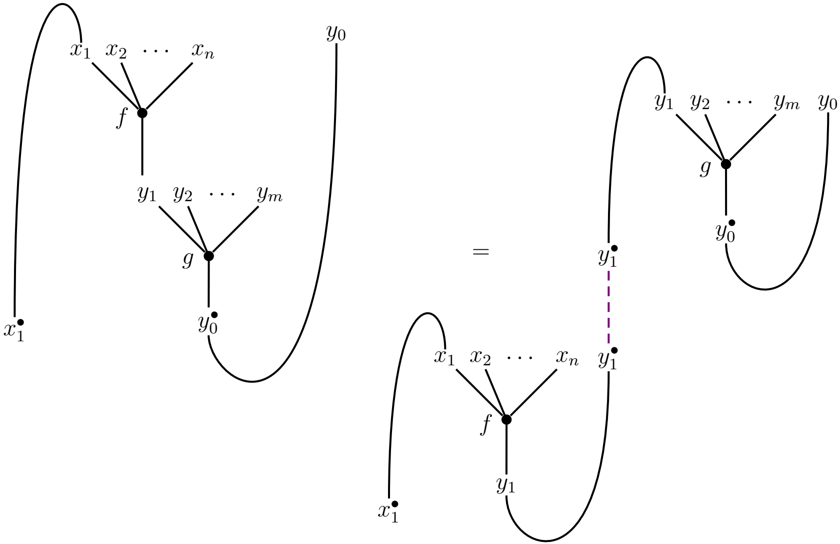
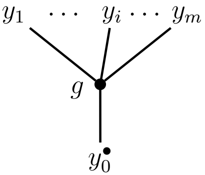
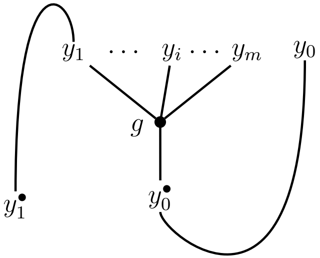




y_1
$\cdots$
y_i
$\cdots$
y_m
g
y_0



y_1
$\cdots$
y_i
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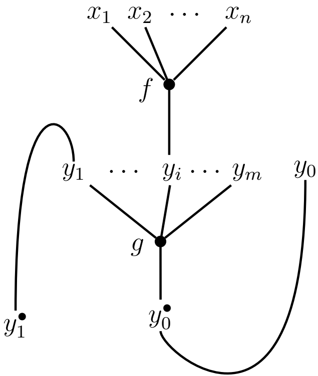
y_0
g
$y_1^\bullet$
$y_0^\bullet$


x_1
x_2
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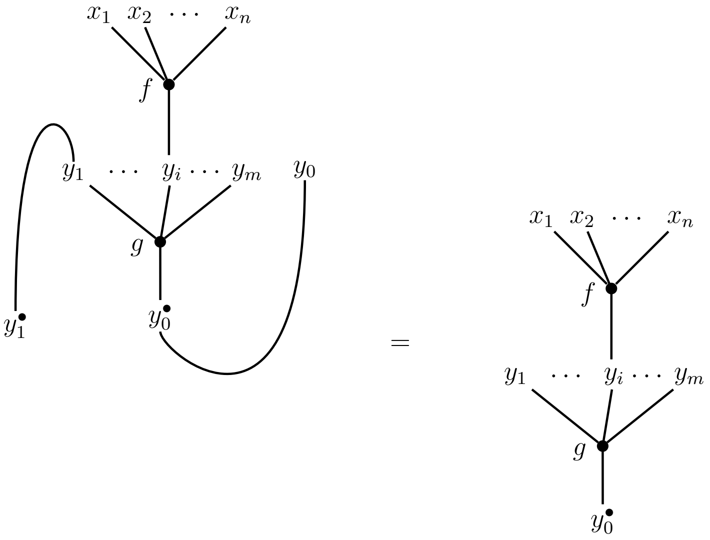
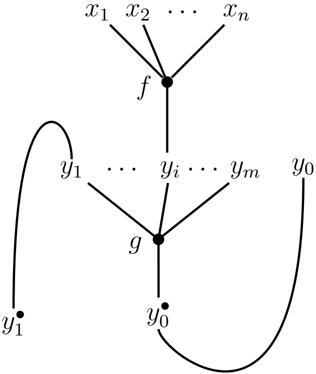
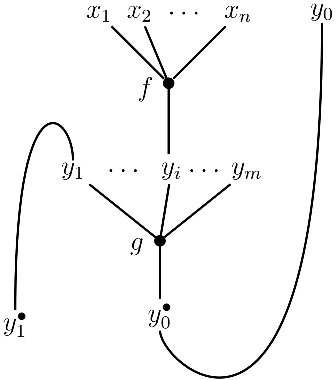



x_1 x_2 $\cdots$ x_n
f
y_1 $\cdots$ y_i $\cdots$ y_m y_0
g
$y_1^\bullet$
$y_0^\bullet$

$=$


x_1 x_2 $\cdots$ x_n y_0
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Example of multicategory

- objects: categories
- multimaps $(A_1, \ldots, A_n) \longrightarrow A_0$: multivariable functors i.e. functors $A_1 \times \cdots \times A_n \longrightarrow A_0$

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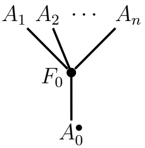

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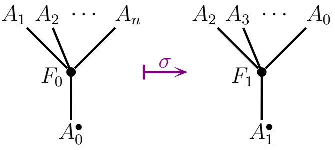

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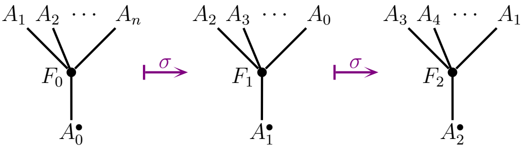

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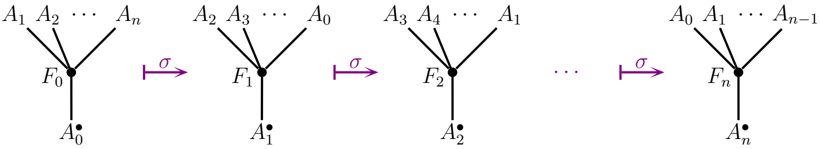

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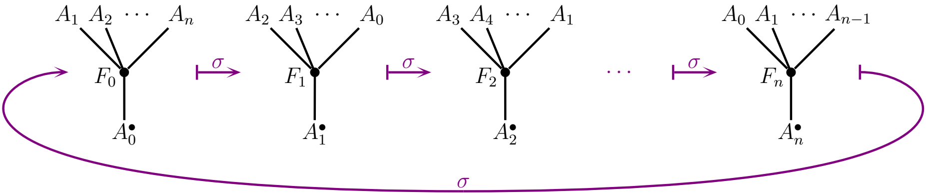

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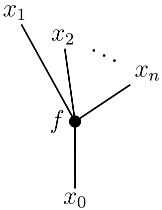
- vertical multi-maps



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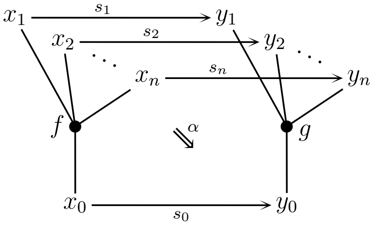

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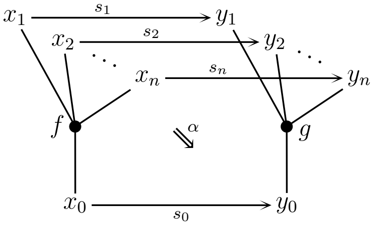

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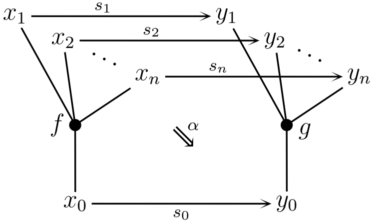

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- on 2-cells:

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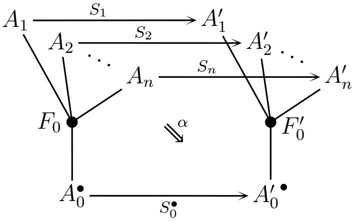

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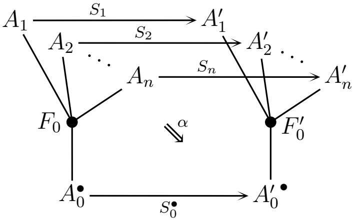

$$\begin{tikzcd}
A_1 \times \cdots \times A_n \arrow[rr, "S_1 \times \cdots \times S_n"] \arrow[d, "F_0"'] \arrow[drr, phantom, "\Nearrow\alpha"] & & A'_1 \times \cdots \times A'_n \arrow[d, "F'_0"] \\
A_0^\bullet \arrow[rr, "S_0^\bullet"'] & & A'^\bullet_0
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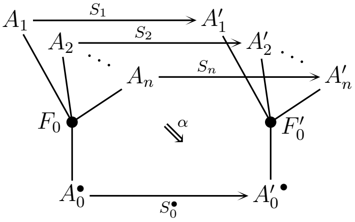

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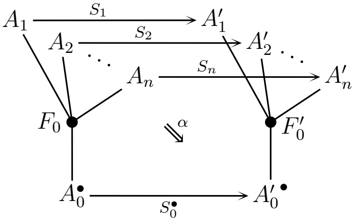

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Cyclic action on 2-cells: parametrised mates correspondence.

Axioms: functoriality of the mates correspondence.

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- natural transformations λ and ρ inducing
  1. a colax comonad map $L_1 \longrightarrow L_2$ from (F, λ)
  2. a lax monad map $R_2 \longrightarrow R_1$ from (G, ρ)

  where λ and ρ are related by mates.

4. Application

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