Fibrations in tricategories

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Centre for Mathematical Sciences
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Fibrations of categories

A cartesian morphism with respect to a functor

Definition

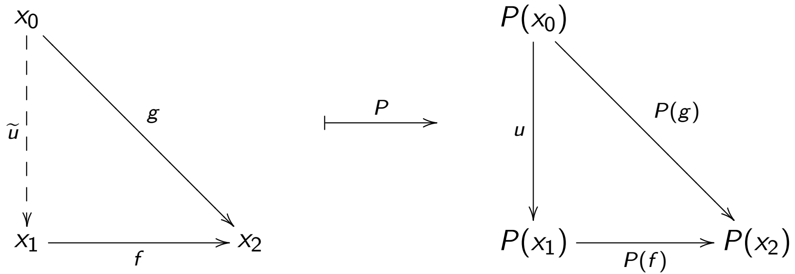
Let $P\colon \mathcal{E} \to \mathcal{B}$ be a functor. A morphism $f\colon x_1 \to x_2$ in $\mathcal{E}$ is cartesian if the diagram

$$
\begin{array}{ccc}
\mathcal{E}(x_0, x_1) & \xrightarrow{\mathcal{E}(x_0,f)} & \mathcal{E}(x_0, x_2) \\
{\scriptstyle P_{x_0,x_1}} \downarrow & & \downarrow {\scriptstyle P_{x_0,x_2}} \\
\mathcal{B}(P(x_0), P(x_1)) & \xrightarrow[\mathcal{B}(P(x_0),P(f))]{} & \mathcal{B}(P(x_0), P(x_2))
\end{array}
$$

is a pullback in *Set*.

A universal property of the cartesian morphism

For any 1-morphism $g\colon x_0 \to x_2$ in $\mathcal{E}$ and any 1-morphism $u\colon P(x_0) \to P(x_1)$ in $\mathcal{B}$, such that $P(g) = P(f) \circ u$



there exists a unique $\widetilde{u}\colon x_0 \to x_1$ such that $g = f \circ \widetilde{u}$.

Fibration of categories - definition

Definition

A functor $P\colon \mathcal{E} \to \mathcal{B}$ has enough cartesian morphisms if for any object x_1 in $\mathcal{E}$ and any morphism $u\colon y_0 \to P(x_1)$ in $\mathcal{B}$, there exists a cartesian morphism $\widetilde{u}\colon x_0 \to x_1$ in $\mathcal{E}$ such that $P(\widetilde{u}) = u$.

Definition

A functor $P\colon \mathcal{E} \to \mathcal{B}$ is called a *fibration of categories* (or a *fibered category*) if it has enough cartesian morphisms.

Properties of fibrations of categories

Theorem

- *Fibrations of categories are closed under composition in Cat.*

Properties of fibrations of categories

Theorem

- *Fibrations of categories are closed under composition in* Cat.
- *Fibrations of categories are closed under pullback in* Cat.

Fibrations in 2-categories

Representable definition

One possibility is simply to *lift* the original definition, by defining a 1-morphism $p\colon E \to B$ in a 2-category to be a fibration if, for every object A in $\mathcal{K}$, the induced functor $p_*\colon \mathcal{K}(A, E) \to \mathcal{K}(A, B)$ is a fibration.

Cartesian 2-morphisms in a 2-category

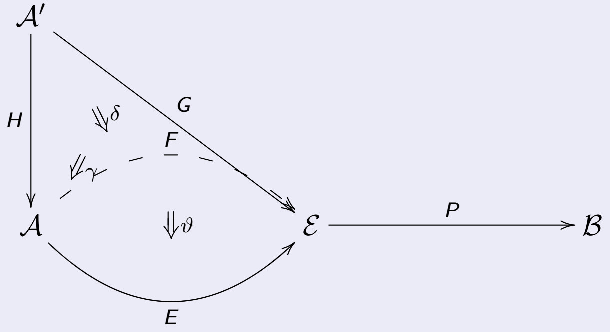
The class of cartesian morphisms in a general 2-category?

Such definition would require the identification of the class of cartesian morphisms in a general 2-category, i.e. those 2-morphisms which occur as *good* liftings of 2-morphisms in the base.

Cartesian 2-morphisms in a 2-category

Definition

A 2-morphism $\vartheta\colon F \Rightarrow E$ in a 2-category $\mathcal{K}$ is cartesian w.r.t. P

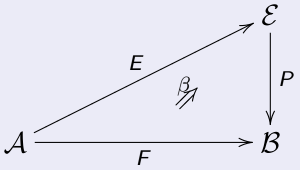


if for any $\gamma\colon G \Rightarrow EH$ and $\xi\colon PG \Rightarrow PFH$ there exists a unique $\widetilde{\xi}\colon G \Rightarrow FH$, such that $P\widetilde{\xi} = \xi$ and $\gamma = (\vartheta H)\widetilde{\xi}$.

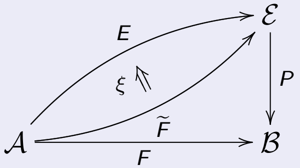
The representable definition of fibrations in a 2-category

Definition

A 1-morphism $p \colon E \to B$ in a 2-category $\mathcal{K}$ is a fibration



if for any $\beta \colon F \Rightarrow PE$ t.e. a cartesian $\xi \colon E \Rightarrow \widetilde{F}$ s.t. $\beta = P\xi$.



Equivalence between representable and original fibrations

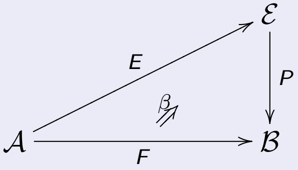
The theorem is present at least in the work of Gray *(Fibred and cofibred categories)*

Theorem

A functor $P\colon \mathcal{E} \to \mathcal{B}$ is a fibration of categories if and only if it is a fibration in the 2-category Cat.

Proof.

1. Necessity - enough to take a terminal category for $\mathcal{A}$ in a diagram



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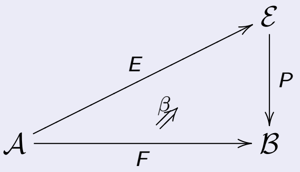


2. Sufficiency - slightly more involved

Semi (op)lax adjunctions

Known as weak quasi-adjunctions in the work of Gray *(Formal Category Theory)*

Definition

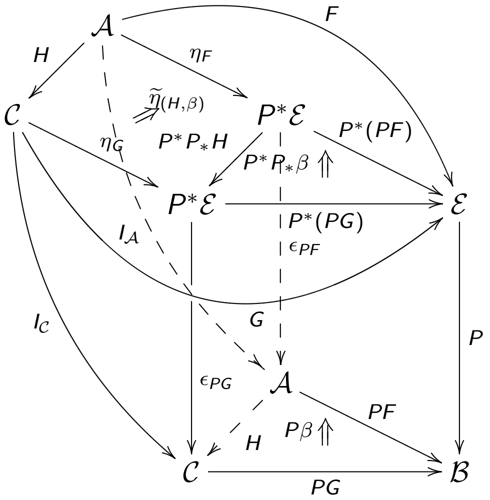
We say that the functor $G\colon \mathcal{K} \to \mathcal{L}$ is a semi-(op)lax right adjoint to $F\colon \mathcal{L} \to \mathcal{K}$ if we are given a (pseudo-) natural transformation $\epsilon\colon FG \Rightarrow Id_{\mathcal{L}}$ and an (op)lax natural transformation $\eta\colon Id_{\mathcal{K}} \Rightarrow GF$ which satisfy the usual triangular identities.

Fibrations in 2-categories - Johnstone's definition

Definition

A 1-morphism $p\colon E \to B$ in a 2-category $\mathcal{K}$ is a fibration if the functor $\Sigma_p\colon \mathcal{K} \swarrow E \to \mathcal{K} \swarrow B$ has a semi-oplax right adjoint $\hat{p}\colon \mathcal{K} \swarrow B \to \mathcal{K} \swarrow E$, such that the semi-oplax adjunction restricts to a pseudo adjunction between corresponding pseudo slice 2-categories K/E and K/B. We say that $p\colon E \to B$ is a strict fibration if the semi-oplax adjunction restricts to a strict adjunction between corresponding strict slice 2-categories $K/^s E$ and $K/^s B$.

A unit oplax natural transformation



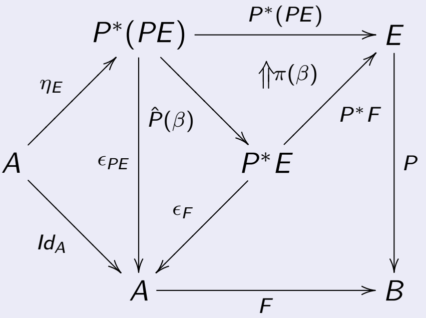
Any Johnstone's fibration is a representable fibration

The theorem is due to Johnstone *(Fibrations and Partial Products in a 2-Category)*

Theorem

Let a 1-morphism $p\colon E \to B$ in $\mathcal{K}$ be a fibration in the sense of Johnstone. Then it is a representable fibration in the 2-category $\mathcal{K}$.

Proof.



Any representable fibration is Johnstone's fibration

The theorem is due to Johnstone *(Fibrations and Partial Products in a 2-Category)*

Theorem

If $\mathcal{K}$ is a bicategory with bipullbacks and $p\colon E \to B$ in $\mathcal{K}$ is a representable fibration, then it is a fibration in the sense of Johnstone.

Fibrations of bicategories

Cartesian 1-morphisms

Definition

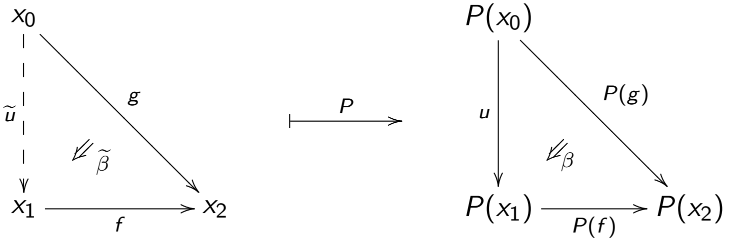
Let $P\colon \mathcal{E} \to \mathcal{B}$ be a strict homomorphism of bicategories. A 1-morphism $f\colon x_1 \to x_2$ in $\mathcal{E}$ is cartesian if the diagram

$$\begin{array}{ccc}
\mathcal{E}(x_0,x_1) & \xrightarrow{\mathcal{E}(x_0,f)} & \mathcal{E}(x_0,x_2) \\
\downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P} \\
\mathcal{B}(P(x_0),P(x_1)) & \xrightarrow[\mathcal{B}(P(x_0),P(f))]{} & \mathcal{B}(P(x_0),P(x_2))
\end{array}$$

is a bicomma object in Cat.

1-Cartesian 1-morphisms

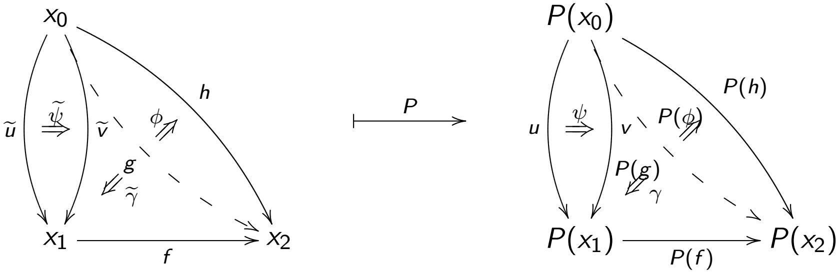
For any 1-morphism $g\colon x_0 \to x_2$ in $\mathcal{E}$, and any 1-morphism $u\colon P(x_0) \to P(x_1)$ and 2-morphism $\beta\colon P(g) \Rightarrow P(f) \circ u$ in $\mathcal{B}$



there exists a 1-morphism $\widetilde{u}\colon x_0 \to x_1$ and a 2-isomorphism $\widetilde{\beta}\colon g \Rightarrow f \circ \widetilde{u}$ in $\mathcal{E}$, such that $P(\widetilde{u}) = u$ and $P(\widetilde{\beta}) = \beta$. We say that the 1-morphism $f\colon x_1 \to x_2$ is a **1-cartesian 1-morphism**, and we call a pair $(\widetilde{u}, \widetilde{\beta})$ a **lifting** of (u, β) by f along g.

2-Cartesian 1-morphisms

For any 2-morphism $\phi\colon g \Rightarrow h$ in $\mathcal{E}$, and any 2-morphism $\psi\colon u \Rightarrow v$ in $\mathcal{B}$, such that $(P(f) \circ \psi)\beta = \gamma P(\phi)$ for some 2-morphism $\gamma\colon P(h) \Rightarrow P(f) \circ v$



there exists a unique 2-morphism $\widetilde{\psi}\colon \widetilde{u} \Rightarrow \widetilde{v}$, such that $P(\widetilde{\psi}) = \psi$ and $(f \circ \widetilde{\psi})\widetilde{\beta} = \widetilde{\gamma}\phi$. We say that the 1-morphism $f\colon x_1 \to x_2$ is a **2-cartesian 1-morphism**, and we call a 2-morphism $\widetilde{\psi}$ a **lifting** of a 2-morphism ψ by f along ϕ.

When there are enough 2-cartesian 1-morphism?

Definition

A homomorphism $P\colon \mathcal{E} \to \mathcal{B}$ has enough cartesian morphisms if for any object x_1 in $\mathcal{E}$ and any 1-morphism $u\colon y_0 \to P(x_1)$ in $\mathcal{B}$, there exists a 2-cartesian 1-morphism $\widetilde{u}\colon x_0 \to x_1$ in $\mathcal{E}$ s.t. $P(\widetilde{u}) = u$.

Fibration of bicategories - definition

Definition

A strict homomorphism of bicategories $P\colon \mathcal{E} \to \mathcal{B}$ is called a 2-fibration if the following conditions are satisfied:

- there are enough 2-cartesian 1-morphisms

Fibration of bicategories - definition

Definition

A strict homomorphism of bicategories $P\colon \mathcal{E} \to \mathcal{B}$ is called a 2-fibration if the following conditions are satisfied:

- there are enough 2-cartesian 1-morphisms
- a homomorphism $P\colon \mathcal{E} \to \mathcal{B}$ is locally a fibration

Fibration of bicategories - definition

Definition

A strict homomorphism of bicategories $P\colon \mathcal{E} \to \mathcal{B}$ is called a 2-fibration if the following conditions are satisfied:

- there are enough 2-cartesian 1-morphisms
- a homomorphism $P\colon \mathcal{E} \to \mathcal{B}$ is locally a fibration
- for any 1-morphism $f\colon x \to y$ in $\mathcal{E}$ a Cartesian functor

$$\begin{array}{ccc}
\mathcal{E}(y,z) & \xrightarrow{\mathcal{E}(f,z)} & \mathcal{E}(x,z) \\
{\scriptstyle P_{y,z}}\downarrow & & \downarrow{\scriptstyle P_{x,z}} \\
\mathcal{B}(P(y),P(z)) & \xrightarrow[\mathcal{B}(P(f),P(z))]{} & \mathcal{B}(P(x),P(z))
\end{array}$$

Properties of fibrations of bicategories

Theorem

- *Fibrations of bicategories are closed under composition in* $\mathcal{B}icat$.

Properties of fibrations of bicategories

Theorem

- *Fibrations of bicategories are closed under composition in Bicat.*
- *Fibrations of bicategories are closed under pullback in Bicat.*

Examples of fibrations of bicategories

Example (Grothendieck fibrations)

Any functor $P\colon \mathcal{E} \to \mathcal{B}$ between categories may be seen as a strict homomorphism of locally discrete bicategories. Then it follows from the conjunction of the two defining properties of cartesian 1-morphisms that the functor P is a 2-fibration of (locally discrete) bicategories if and only if it is a Grothendieck fibration of categories.

Examples of fibrations of bicategories

Example (Fibrations of groupoids)

Any fibration $P\colon \mathcal{H} \to \mathcal{G}$ of groupois, introduced by Brown, is a special case of a Grothendieck fibration of categories, and therefore it may be seen as a 2-fibration of (locally discrete) bicategories as in the previous example. From such fibrations, Brown derived a family of exact sequences familiar in homotopy theory, including a six term exact sequences familiar in nonabelian cohomology, which naturally led to the definition of a nonabelian cohomolgy of groupoids with coefficients in groupoids.

Examples of fibrations of bicategories

Example (Fibrations of 2-groupoids)

The category $2Gpd_{str}$ of 2-groupoids and their strict homomorphisms have a closed model structure, described by Moerdijk and Svensson, and its homotopy category is equivalent to the homotopy category of a closed model structure on the category $2\text{-}\mathcal{G}pd$ of 2-groupoids and their homomorphisms. A strict homomorphisms of 2-groupoids is a fibration in the model structure on $2Gpd_{str}$ if and only if it is a 2-fibration of bicategories.

Examples of fibrations of bicategories

Example (Fibrations of bigroupoids)

Fibration of bigroupoids, introduced by Hardie, Kamps and Kieboom, generalized the notion of fibration of 2-groupoids by Moerdijk from the previous example. They used Brown's construction in order to derive an exact nine term sequence from such fibrations, and they applied their theory to the construction of a homotopy bigroupoid of a topological space

Examples of fibrations of bicategories

Example (Fibrations in model structures on 2-*Cat* and $Bicat_s$)

The category 2-*Cat* of 2-categories and 2-functors has a closed model structure, introduced by Lack, which he extended to the category $Bicat_s$ of bicategories and their strict homomorphisms. These model structures are closely related to the model structure of Moerdijk and Svensson. Fibrations in these model categories are strict homomorphisms having the *equivalence lifting property*. Therefore, 2-fibrations of bicategories are special cases of fibrations in closed model structures on categories 2-*Cat* and $Bicat_s$, as those strict homomorphisms $F\colon \mathcal{A} \to \mathcal{B}$ having the lifting property for all 1-morphisms in $\mathcal{A}$, and not just for equivalences.

Examples of fibrations of bicategories

Example (A domain fibration of the homotopy fiber of a bicategory)

Examples of fibrations of bicategories

Example (A codomain fibration of the bicategory with finite bilimits)

Examples of fibrations of bicategories

Example (Shulman's monoidal fibrations)

Examples of fibrations of bicategories

Example (Zawadowski's lax monoidal fibrations)

Fibrations in tricategories

Representable definition of fibrations in tricategories

Representable definition

Again, one possibility is to *lift* the definition of fibrations of bicategories, by defining a 1-morphism $p\colon E \to B$ in a tricategory to be a fibration if, for every object A in $\mathcal{K}$, the induced homomorphism $p_*\colon \mathcal{K}(A, E) \to \mathcal{K}(A, B)$ is a fibration of bicategories.

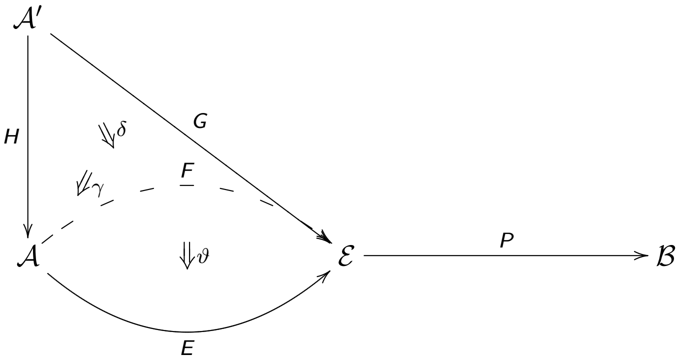
Cartesian 2-morphisms in a tricategory

The class of cartesian morphisms in a general tricategory?

Again, such definition would require the identification of the class of cartesian morphisms in a general tricategory, i.e. those 3-morphisms which occur as *good* liftings of 3-morphisms in the base.

Cartesian 1-morphisms in a tricategory

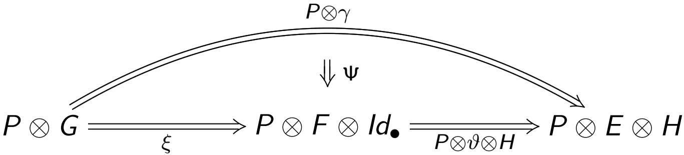
A 2-morphism $\vartheta\colon F \Rightarrow E$ in *Bicat* is 1-cartesian w.r.t. $P\colon \mathcal{E} \to \mathcal{B}$



if for any $\gamma\colon G \Rightarrow E \otimes H$ and $\xi\colon P \otimes G \Rightarrow P \otimes F \otimes H$

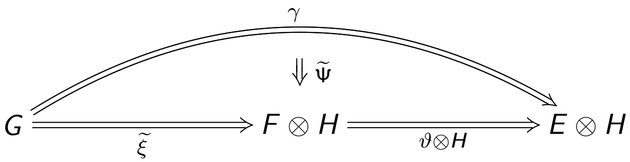
Cartesian 1-morphisms in a tricategory

and for any modification $\Psi\colon P \otimes \gamma \Rrightarrow (P \otimes \vartheta \otimes H) \circ \xi$



Cartesian 1-morphisms in a tricategory

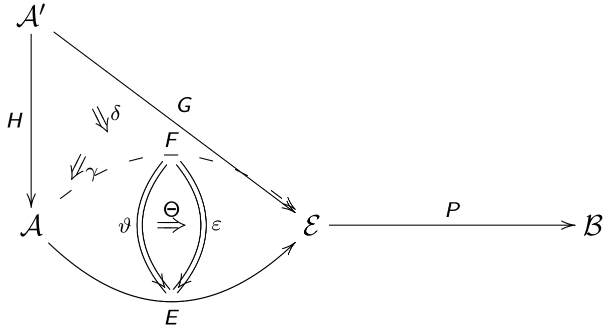
there exists a pseudonatural transformation $\widetilde{\xi}\colon G \Rightarrow F \otimes H$, together with a modification $\widetilde{\Psi}\colon \gamma \Rrightarrow (\vartheta \otimes H) \circ \widetilde{\xi}$



such that $P \otimes \widetilde{\xi} = \xi$ and $P \otimes \widetilde{\Psi} = \Psi$.

Cartesian 3-morphisms in a tricategory

A 3-morphism $\Theta\colon \vartheta \Rrightarrow \varepsilon$ in *Bicat* is 1-cartesian w.r.t. $P\colon \mathcal{E} \to \mathcal{B}$



for any $\Omega\colon P \otimes \gamma \Rrightarrow P \otimes [(\vartheta \otimes H) \circ \delta]$ and $\Sigma\colon \gamma \Rrightarrow (\varepsilon \otimes H) \circ \delta$ such that

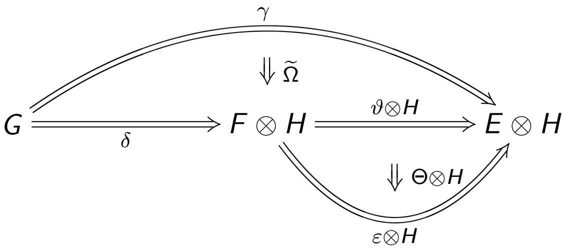
$$P \otimes \Sigma = \{P \otimes [(\Theta \otimes H) \circ \delta]\}\Omega$$

Cartesian 3-morphisms in a tricategory

there exist a unique modification $\widetilde{\Omega}\colon \gamma \Rrightarrow (\vartheta \otimes H) \circ \delta$

$$\Omega = P \otimes \widetilde{\Omega}$$

and $\Sigma = [(\Theta \otimes H) \circ \delta]\widetilde{\Omega}$ is equal to the vertical composition

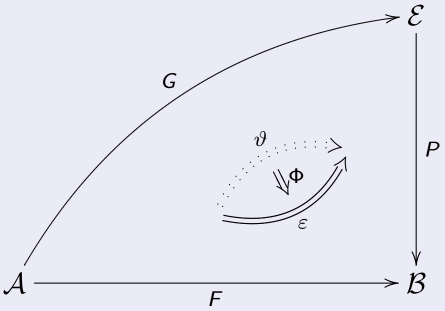


in the bicategory $Bicat(\mathcal{A}', \mathcal{E})$.

Fibrations in a tricategory

Definition

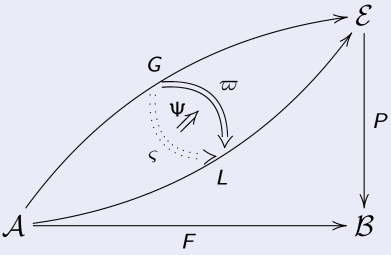
A 1-morphism $P\colon \mathcal{E} \to \mathcal{B}$ in *Bicat* is a fibration if for every 3-morphism $\Phi\colon \vartheta \Rrightarrow \varepsilon$



Fibrations in a tricategory

Definition

there exists a cartesian 3-morphism $\Psi\colon \varsigma \Rrightarrow \varpi$



such that

$$P \otimes \Psi = \Phi$$

Equivalence between representable and original fibrations

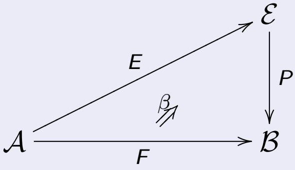
The theorem is present at least in the work of Gray *(Fibred and cofibred categories)*

Theorem

A homomorphism $P\colon \mathcal{E} \to \mathcal{B}$ is a fibration of bicategories if and only if it is a fibration in the tricategory Bicat.

Proof.

1. Necessity - enough to take a terminal category for $\mathcal{A}$ in a diagram



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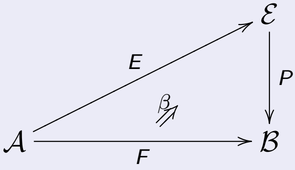


2. Sufficiency - slightly more involved

□

Semi (op)lax 3-adjunctions

Known as weak quasi-adjunctions in the work of Gray *(Formal Category Theory)*

Definition

We say that the trihomomorphism $G\colon \mathcal{K} \to \mathcal{L}$ is a semi-(op)lax right 3-adjoint to $F\colon \mathcal{L} \to \mathcal{K}$ if we are given a (pseudo-) natural tritransformation $\epsilon\colon FG \Rightarrow Id_{\mathcal{L}}$ and an (op)lax natural tritransformation $\eta\colon Id_{\mathcal{K}} \Rightarrow GF$ which satisfy coherence with respect to triangulators.

Fibrations in tricategories - intrinsic definition

Definition

A 1-morphism $p\colon E \to B$ in a tricategory $\mathcal{K}$ is a fibration if the trihomomorphism $\Sigma_p\colon \mathcal{K} \swarrow E \to \mathcal{K} \swarrow B$ has a semi-oplax right adjoint $\hat{p}\colon \mathcal{K} \swarrow B \to \mathcal{K} \swarrow E$, such that the semi-oplax 3-adjunction restricts to a pseudo 3-adjunction between corresponding pseudo slice tricategories K/E and K/B. We say that $p\colon E \to B$ is a strict fibration if the semi-oplax 3-adjunction restricts to a strict 3-adjunction between corresponding strict slice tricategories $K/^sE$ and $K/^sB$.

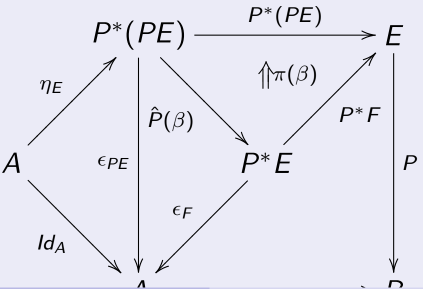
Any intrinsic fibration in a tricategory is a representable fibration

The work in progress *(Fibrations of bicategories)*

Theorem

Let a 1-morphism $p\colon E \to B$ in $\mathcal{K}$ be a fibration in the sense from previous definition. Then it is a representable fibration in $\mathcal{K}$.

Proof.



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