Generalizing canonical extension to the categorical setting

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Outline

1. Introduction to duality theory and canonical extension
2. Semantics for coherent first order logic ($\wedge$, $\vee$, $\bot$, $\top$, $\exists$):
   - Coherent categories
   - Coherent hyperdoctrines
3. Canonical extension in the categorical setting
4. Relate to Makkai's topos of types

Stone duality

Boolean algebras: structures $(B, \wedge, \vee, \neg, 0, 1)$.

Boolean spaces: compact, totally disconnected, Hausdorff spaces.

Boolean algebras	$\leftrightarrows$	**Boolean spaces**
$Cl(X)$	$\leftarrow\!\shortmid$	X
B	$\mapsto$	$(PrIdl(B), \tau_B)$

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Stone Representation Theorem: every Boolean algebra is embeddable in a powerset algebra.

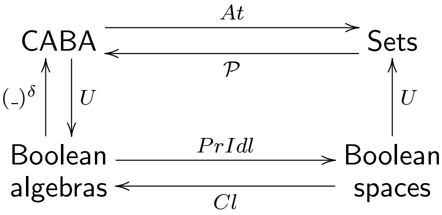
Proof: for a Boolean algebra B,

$$B \cong Cl(PrIdl(B)) \hookrightarrow \mathcal{P}(PrIdl(B)).$$

Stone duality and canonical extension

Canonical extension: algebraic description of topological duality.

Study $B \cong Cl(PrIdl(B)) \hookrightarrow \mathcal{P}(PrIdl(B)) = B^\delta$.

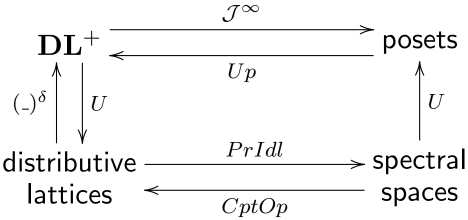


CABA	=	complete and atomic Boolean algebras.
Boolean spaces	=	compact, totally disconnected Hausdorff spaces.

Canonical extension of distributive lattices

Canonical extension: algebraic description of topological duality.

Study $L \cong CptOp(PrIdl(L)) \hookrightarrow Up(PrIdl(L)) = L^\delta$.



$\mathbf{DL}^+$ = completely distributive algebraic lattices.

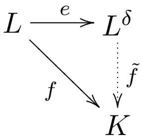
spectral spaces = sober spaces with a basis of compact opens.

Canonical extension of distributive lattices

$\mathbf{DL}^+$ = completely distributive algebraic lattices.

Canonical extension is left adjoint to $\mathbf{DL}^+ \hookrightarrow \mathbf{DL}$.

Universal characterization of canonical extension:



where $L \in \mathbf{DL}$ and $K, L^\delta \in \mathbf{DL}^+$.

Semantics for coherent logic

Coherent logic = fragment of first order logic in $\wedge, \vee, \bot, \top, \exists$.

A **coherent category** is a category $\mathbf{C}$ satisfying

1. $\mathbf{C}$ has finite limits;
2. $\mathbf{C}$ has stable finite unions;
3. $\mathbf{C}$ has stable images.

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Remark: all subobject posets are distributive lattices.

Idea: apply canonical extension to those separately.

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A **coherent hyperdoctrine** is a functor $P \colon \mathbf{B}^{op} \to \mathbf{DL}$ s.t.

1. $\mathbf{B}$ has finite limits;
2. for all $A \xrightarrow{\alpha} B$ in $\mathbf{B}$, $P(\alpha)$ has a left adjoint satisfying Frobenius and Beck-Chevalley.

Coherent categories and coherent hyperdoctrines

Proposition: there is a 2-categorical adjunction

$$\mathcal{A}\colon \mathbf{CHyp} \leftrightarrows \mathbf{Coh}\colon \mathcal{S},$$

where $\mathcal{A} \dashv \mathcal{S}$ and $\mathcal{A}(\mathcal{S}(\mathbf{C})) \simeq \mathbf{C}$.

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For $\mathbf{C} \in \mathbf{Coh}$, $\quad \mathcal{S}(\mathbf{C}) = \mathcal{S}_{\mathbf{C}}\colon \begin{array}{rcl} \mathbf{C}^{op} & \to & \mathbf{DL} \\ A & \mapsto & Sub_{\mathbf{C}}(A) \end{array}$

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For $P\colon \mathbf{B}^{op} \to \mathbf{DL}$, $\mathcal{A}(P)$ is the category with:

objects are pairs (A, a), where $A \in \mathbf{B}$, $a \in P(A)$;

a morphism $(A, a) \to (B, b)$ is an element $f \in P(A \times B)$ which is a functional relation $(A, a) \to (B, b)$.

Canonical extension of coherent hyperdoctrines

Recall: canonical extension for DL's is a functor $\mathbf{DL} \xrightarrow{(\_)^\delta} \mathbf{DL}^+$.

Definition

For a coh. hyperdoctrine $P\colon \mathbf{B}^{op} \to \mathbf{DL}$ we define:

$$P^\delta\colon \mathbf{B}^{op} \xrightarrow{P} \mathbf{DL} \xrightarrow{(\_)^\delta} \mathbf{DL}.$$

Proposition

For a coh. hyperdoctrine P, P^δ is again a coh. hyperdoctrine.

Proof: check that, for all $A \xrightarrow{\alpha} B$ in $\mathbf{B}$, $P^\delta(\alpha)$ has a left adjoint satisfying BC and Frobenius.

Canonical extension of coherent categories

We have:

- adjunction $\mathcal{A}\colon \mathbf{CHyp} \leftrightarrows \mathbf{Coh}\colon \mathcal{S}$, $\mathbf{C} \simeq \mathcal{A}(\mathcal{S}_{\mathbf{C}})$;
- for $P \in \mathbf{CHyp}$, $P^{\delta}\colon \mathbf{B}^{op} \xrightarrow{P} \mathbf{DL} \xrightarrow{(\_)^{\delta}} \mathbf{DL}$.

Definition

For a coherent category $\mathbf{C}$ we define:

$$\mathbf{C}^{\delta} = \mathcal{A}(\mathcal{S}^{\delta}_{\mathbf{C}}).$$

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Definition

For a coherent category $\mathbf{C}$ we define:

$$\mathbf{C}^\delta = \mathcal{A}(\mathcal{S}^\delta_{\mathbf{C}}).$$

Proposition

For a distributive lattice $\mathbf{L}$, $\mathcal{A}(\mathcal{S}^\delta_{\mathbf{L}}) \simeq \mathbf{L}^\delta$.

Canonical extension of coherent categories

Properties of $\mathbf{C}^\delta = \mathcal{A}(\mathcal{S}^\delta_{\mathbf{C}})$:

1. subobject lattices are in $\mathbf{DL}^+$;
2. pullback morphisms are complete lattice homomorphisms.

$\mathbf{Coh}^+$ = coherent categories satisfying (1) and (2).

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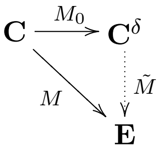
Universal characterization:



where $\mathbf{C} \in \mathbf{Coh}$, $\mathbf{E}, \mathbf{C}^\delta \in \mathbf{Coh}^+$, M a coherent functor satisfying:

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Universal characterization:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{M_0} & \mathbf{C}^\delta \\ & {\scriptstyle M}\searrow & \downarrow{\scriptstyle \tilde{M}} \\ & & \mathbf{E} \end{array}$$

where $\mathbf{C} \in \mathbf{Coh}$, $\mathbf{E}, \mathbf{C}^\delta \in \mathbf{Coh}^+$, M a coherent functor satisfying:

for all $A \xrightarrow{\alpha} B$ in $\mathbf{C}$, ρ (prime) filter in $\mathcal{S}_C(A)$,

$\exists_{M(\alpha)}(\bigwedge\{M(U) \mid U \in \rho\}) \cong \bigwedge\{\exists_{M(\alpha)}(M(U)) \mid U \in \rho\}$.

Canonical extension of Heyting categories

Heyting categories provide semantics for first order logic.

Canonical extension interacts well with Heyting structure:

- for a coherent category $\mathbf{C}$, $\mathbf{C}^\delta$ is a Heyting category;
- for a morphism of Heyting categories $F\colon \mathbf{C} \to \mathbf{D}$,

$$F^\delta\colon \mathbf{C}^\delta \to \mathbf{D}^\delta$$

  is again a morphism of Heyting categories.

Topos of types

Topos of types was introduced by Makkai in 1979 as:

- ‘a reasonable codification of the ‘discrete’ (non topological) syntactical structure of types of the theory’,
- a tool to prove representation theorems,
- ‘conceptual tool meant to enable us to formulate precisely certain natural intuitive questions’.

Some later work by: Magnan & Reyes and Butz.

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Alternative construction:

The functor $\mathcal{S}^{\delta}_{\mathbf{C}} \colon \mathbf{C}^{op} \to \mathbf{DL}$ is an internal frame in $Sh(\mathbf{C}, J_{coh})$.

Then $Sh(\mathcal{S}^{\delta}_{\mathbf{C}}) \simeq T(\mathbf{C}) =$ topos of types of $\mathbf{C}$.

Topos of types and morphisms

Theorem: for a coherent functor $F\colon \mathbf{C} \to \mathbf{D}$,

- if F is conservative, then $T(F)\colon T(\mathbf{D}) \to T(\mathbf{C})$ is a geometric surjection;
- if F is a morphism of Heyting categories, then $T(F)\colon T(\mathbf{D}) \to T(\mathbf{C})$ is open.

Topos of types and the class of models

For a distributive lattice L,

$$\begin{aligned}\text{prime ideals of } L &= \text{lattice homomorphisms } L \to \mathbf{2} \\ &= \text{'models of } L\text{'.}\end{aligned}$$

$L^\delta = Up(Mod(L))$.

Categorical analogue:

$Mod(\mathbf{C})$ = coherent functors $M\colon \mathbf{C} \to \mathbf{Set}$.

Study: $\mathbf{Set}^{Mod(\mathbf{C})}$.

We have to restrict to an appropriate subcategory $\mathcal{K}$ of $Mod(\mathbf{C})$.

Question: How does $\mathbf{Set}^{\mathcal{K}}$ relate to $T(\mathbf{C}) = Sh(\mathcal{S}^\delta_{\mathbf{C}})$?

Topos of types and the class of models

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Evaluation functor

$$\begin{array}{rcl} ev\colon \mathbf{C} & \to & \mathbf{Set}^{\mathcal{K}} \\ A & \mapsto & \begin{array}{rcl} ev(A)\colon \mathcal{K} & \to & \mathbf{Set} \\ M & \mapsto & M(A) \end{array} \end{array}$$

Gives a geometric morphism $\phi_{ev}\colon \mathbf{Set}^{\mathcal{K}} \to Sh(\mathbf{C}, J_{coh})$.

Topos of types and the class of models

Question: How does $\mathbf{Set}^{\mathcal{K}}$ relate to $T(\mathbf{C}) = Sh(\mathcal{S}^{\delta}_{\mathbf{C}})$?

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Gives a geometric morphism $\phi_{ev}\colon \mathbf{Set}^{\mathcal{K}} \to Sh(\mathbf{C}, J_{coh})$.

Theorem: the topos of types yields the hyper-connected localic factorization of $\mathbf{Set}^{\mathcal{K}} \xrightarrow{\phi_{ev}} Sh(\mathbf{C}, J_{coh})$:

$$\begin{array}{ccc} & & T(\mathbf{C}) \\ & \nearrow & \downarrow \\ \mathbf{Set}^{\mathcal{K}} & \xrightarrow[\phi_{ev}]{} & Sh(\mathbf{C}, J_{coh}) \end{array}$$

Future work

We have: notion of canonical extension for coherent categories

We would like to:

- Study the following diagram (where $\mathcal{K} \subseteq Mod(\mathbf{C})$):

$$
\begin{array}{ccc}
 & & T(C) \\
 & \nearrow & \downarrow \\
\mathbf{Set}^{\mathcal{K}} & \xrightarrow[\phi_{ev}]{} & Sh(\mathbf{C}, J_{coh})
\end{array}
$$

- Apply the developed theory in the study of first order logics.
- In particular: study interpolation problems for first order logics, e.g. for IPL $+ (\phi \to \psi) \vee (\psi \to \phi)$.