Relative Frobenius algebras are groupoids

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Introduction

Groupoids:

- ubiquitous: analysis, noncommutative geometry, quantum
- various definitions "groups with more than one object":
  - category whose morphisms are invertible
  - groups with partial multiplication
- generalize to quantum groups / Hopf algebras

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Frobenius algebras:

- algebra, (topological) quantum (field) theory
- definable in dagger monoidal categories (no limits!)

Will make connection precise:

groupoid (in **Set**) = special dagger Frobenius algebra (in **Rel**)

Groupoids

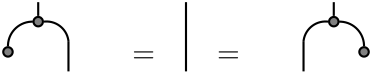
- Internal definition:

$$G_0 \underset{\xleftarrow{\quad s \quad}}{\overset{\xleftarrow{\quad t \quad}}{\xrightarrow{\quad e \quad}}} G_1 \circlearrowleft^{i} \xleftarrow{\quad m \quad} G_2 = G_1 \times_{G_0} G_1$$

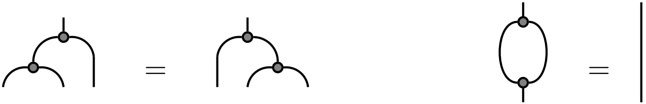
  subject to familiar equations: $m \circ (i \times 1) \circ \Delta = e \circ s$
- Notice: need finite limits in ambient category

Frobenius algebras

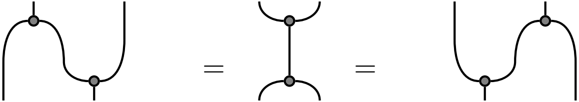
In monoidal category with dagger $\dagger\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$, $X^\dagger = X$, $f^{\dagger\dagger} = f$, $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, consider $m\colon X \otimes X \to X$, $u\colon I \to X$ satisfying:



$m \circ (u \otimes 1) \qquad 1 \qquad m \circ (1 \otimes u)$



$m \circ (m \otimes 1) \qquad m \circ (1 \otimes m) \qquad m \circ m^\dagger \qquad 1$



$(m^\dagger \otimes 1) \circ (1 \otimes m) \qquad m^\dagger \circ m \qquad (1 \otimes m^\dagger) \circ (m \otimes 1)$

Frobenius algebras – why interesting?

[Coecke, Pavlovic, Vicary]:

- commutative Frobenius algebra in **FHilb** = orthonormal basis
- Frob algebra in **FHilb** = fin-dim complex semisimple algebra

But:

- $(X, m, m^\dagger)$ is both Frobenius and Hopf $\implies X \cong I$

Relative Frobenius algebras

Category **Rel**:

- objects are sets
- morphisms $X \to Y$ are relations $R \subseteq X \times Y$
- $S \circ R = \{(x, z) \mid \exists y\colon (x, y) \in R, (y, z) \in S\}$
- $R^\dagger = \{(y, x) \mid (x, y) \in R\}$
- $\otimes$ is cartesian product

Relative Frobenius algebra = Frobenius algebra in **Rel**

[Pavlovic]:

- commutative relative Frobenius algebra
  = disjoint union of abelian groups

From groupoids to relative Frobenius algebras

Given **G**, set

- $X = G_1$
- $m = \{((g, f), g \circ f) \mid s(g) = t(f)\}$ ("graph of multiplication")
- $u = \{(*, e(x)) \mid x \in G_0\}$ ("all identities")

Then:

- $m \circ m^\dagger = \{(f, f) \mid \exists g, h \colon s(h) = t(g), f = h \circ g\} = 1$
- $$\begin{aligned} m^\dagger \circ m &= \{((a, b), (c, d)) \mid a \circ b = c \circ d\} \\ &= \{((a, b), (c, d)) \mid \exists e \colon e \circ d = b, a \circ e = c\} \\ &= (m \times 1) \circ (1 \times m^\dagger) \end{aligned}$$

From relative Frobenius algebras to groupoids

If (X, m, u) is relative Frobenius algebra, then:

- m is single-valued: may write $f = h \circ g$ (when $h \circ g \downarrow$!)
- Frobenius law means
  $a \circ b = c \circ d \iff \exists e \colon b = e \circ d, c = a \circ e$

Set

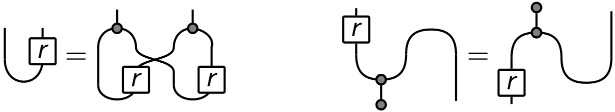
- $G_1 = X$
- $G_2 = \{(g, f) \in X^2 \mid g \circ f \downarrow\}$
- $G_0 = \{x \mid (*, x) \in u\}$
- $e = U \times U$
- $s = \{(f, x) \mid f \circ x \downarrow\}$
- $t = \{(f, y) \mid y \circ f \downarrow\}$
- $i = \{(g, f) \mid g \circ f, f \circ g \in G_0\}$

Theorem: this data forms a (small) groupoid.

Morphisms

Choices of morphisms:

- **Gpd**: functors (natural choice for groupoids)
- $\mathbf{Gpd}^{\mathrm{mfunc}}$: *multi-valued functors*
- $\mathbf{Gpd}^{\mathrm{rel}}$: subgroupoids of $G \times H$ (natural choice for relations)
- $\mathbf{Frob}(\mathbf{Rel})^{\mathrm{rel}}$: relations such that



- **Frob**(**Rel**): relations preserving multiplication and inverses (natural for Frobenius algebras)
- $\mathbf{Frob}(\mathbf{Rel})^{\mathrm{func}}$: functions (relations with a right adjoint) preserving multiplication and inverses
- Equivalences?

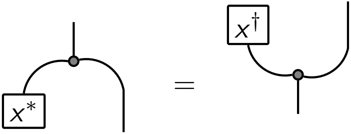
Morphisms

Theorem: constructions are functorial, give isomorphisms:

$$\begin{array}{ccccc} \mathbf{Frob}(\mathbf{Rel})^{\mathrm{func}} & \hookrightarrow & \mathbf{Frob}(\mathbf{Rel}) & \hookrightarrow & \mathbf{Frob}(\mathbf{Rel})^{\mathrm{rel}} \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ \mathbf{Gpd} & \hookrightarrow & \mathbf{Gpd}^{\mathrm{mfunc}} & \hookrightarrow & \mathbf{Gpd}^{\mathrm{rel}} \end{array}$$

H*-algebras

Drop unitality, and replace Frobenius law by its 'square root':
there is involution $*\colon \mathbf{C}(I,X)^{\mathrm{op}} \to \mathbf{C}(I,X)$ with



$m \circ (x^* \otimes 1)$ $\qquad$ $(x^\dagger \otimes 1) \circ m^\dagger$

[Abramsky, Heunen]:

- commutative H*-algebra in **Hilb** = orthonormal basis

Semigroupoids

- "Categories without identities (and inverses)"
- Internal definition:

$$G_0 \underset{s}{\overset{t}{\leftleftarrows}} G_1 \xleftarrow{m} G_2 = G_1 \times_{G_0} G_1$$

- Semifunctors: morphisms $G_i \to G_i'$ commuting with the above

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- *Regular*: each f has *pseudoinverse* f^*: $ff^*f = f$, $f^*ff^* = f^*$
- *Locally cancellative*: $fhh^* = gh^* \implies fh = g$
- Lemma: a locally cancellative regular semigroupoid is a groupoid if and only if it has identities

Thm: relative H*-algebra = locally cancellative regular semigpd

Morphisms

Theorem: there are adjunctions

$$\begin{array}{ccccc}
\mathbf{Hstar(Rel)}^{\mathrm{func}} & \hookrightarrow & \mathbf{Hstar(Rel)} & \hookrightarrow & \mathbf{Hstar(Rel)}^{\mathrm{rel}} \\
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Conclusion

- Can study groupoid-like objects in non-cartesian categories: fundamental uses of groupoids?
- Investigate setting of topological/localic relations; Kleisli category of powerset monad?
- Geometric quantization commutes with reduction: (symplectic) manifolds and (canonical) relations.
- What are groupoids in **FHilb**?