Compact closed bicategories

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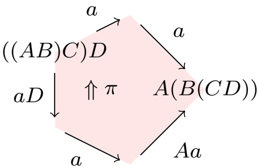
Examples of compact closed bicategories:

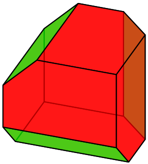
- **Rel** sets, relations, and implications
- **Prof** categories, profunctors, and natural transformations
- **2Vect** 2-vector spaces, linear functors, transformations
- $\boldsymbol{n}\mathbf{Cob_2}$ manifolds, manifolds with boundary, manifolds with corners
- and of course any compact closed category where we take the 2-morphisms to be identities

Compact closed bicategories have...

- a tensor product functor with Stasheff polytopes to govern it

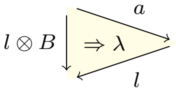
$a : (AB)C \to A(BC)$

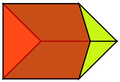




- a monoidal unit with maps of Stasheff polytopes to govern it

$r : A \otimes I \to A$

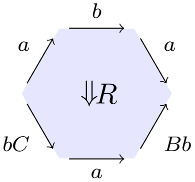


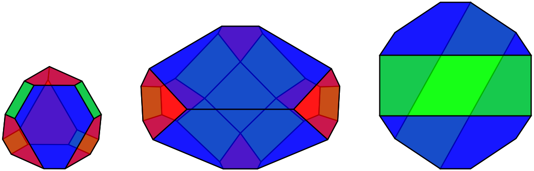


Compact closed bicategories have...

- a braiding with shuffle polytopes and the Breen polytope to govern it

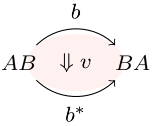
$b : A \otimes B \to B \otimes A$

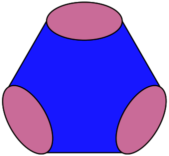


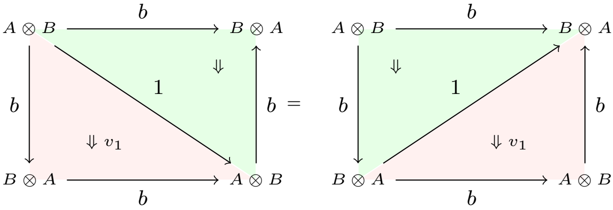


Compact closed bicategories have...

- a syllepsis satisfying equations

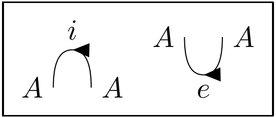


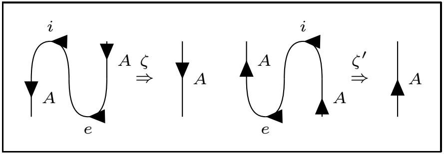


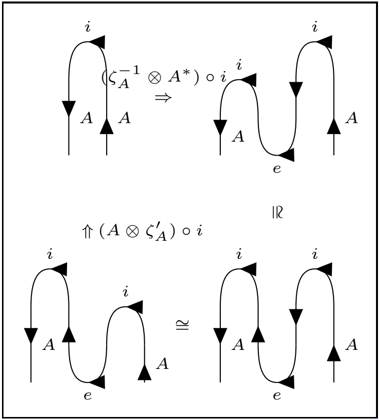


Compact closed bicategories have...

- duals for objects with bends, yanking and the swallowtail law to govern them







Theorem (Stay)

If T is a 2-category with finite products and pseudo pullbacks, then the bicategory $Span_2(T)$ of

- *objects of T*
- *spans in T*
- *isomorphism classes of weak maps of spans*

is compact closed.

Theorem (Stay)

A compact closed bicategory satisfies the axioms of a traced monoidal category in a canonical way up to isomorphism.

We can present the Lawvere theory of a symmetric monoidal category but not of a symmetric monoidal *closed* category because of the currying isomorphism between the contravariant functors

$$\hom(x \otimes y, z) \cong \hom(x, y \multimap z).$$

Conjecture

There exists a compact closed bicategory **Th(SMCC)** such that the 2-category **hom(Th(SMCC), Prof)** of

- symmetric monoidal functors (of bicategories),
- symmetric monoidal natural transformations and
- symmetric monoidal modifications

is 2-equivalent to the 2-category **SMCC** of

- symmetric monoidal closed categories,
- symmetric monoidal closed functors and
- symmetric monoidal closed natural isomorphisms.

Useful fact

Every profunctor with a right adjoint is of the form $\hom(-, F-)$ where F is a functor.

This means that when presenting **Th(SMCC)** we can talk about not only profunctors but also functors. Given a function symbol $F : \mathbf{C} \to \mathbf{D}$ that we mean to be interpreted as a functor, we add

- a function symbol $F' : \mathbf{D} \to \mathbf{C}$,
- 'bend' rewrites $i_F : \mathbf{C} \Rightarrow F' \circ F$ and $e_F : F \circ F' \Rightarrow \mathbf{D}$
- and 'yanking' equations

$$(F \boxtimes e_F) \circ (i_F \boxtimes F) = F$$

$$(e_F \boxtimes F') \circ (F' \boxtimes i_F) = F'$$

Presentation of **Th(SMCC)**, where $\boxtimes$ is the monoidal product, J is the monoidal unit, and $(\cdot)^*$ is the dual.

- a sort C
- function symbols
  - $\otimes \colon C \boxtimes C \to C$
  - $\otimes' \colon C \to C \boxtimes C$
  - $I \colon J \to C$
  - $I' \colon C \to J$
  - $\multimap \colon C^* \boxtimes C \to C$
  - $\multimap' \colon C \to C^* \boxtimes C$

Presentation of **Th(SMCC)**, where $\boxtimes$ is the monoidal product, J is the monoidal unit, and $(\cdot)^*$ is the dual.

- rewrites
  - 'bends' for $\otimes, I$, and $\multimap$
  - $a\colon \otimes \circ (\otimes \boxtimes C) \Rightarrow \otimes \circ (C \boxtimes \otimes) \circ \mathbf{assoc}_{C,C,C}$
  - $b\colon \otimes \Rightarrow \otimes \circ\ \mathbf{swap}_{C,C}$
  - $c : \otimes'^* \Rightarrow \mathbf{curry}^{-1}(\multimap^*)$
  - $l\colon \otimes \circ (I \boxtimes C) \Rightarrow \mathbf{left}_C$
  - $r\colon \otimes \circ (C \boxtimes I) \tilde{\Rightarrow} \mathbf{right}_C$
  - formal inverses for a, b, c, l, r

Presentation of **Th(SMCC)**, where $\boxtimes$ is the monoidal product, J is the monoidal unit, and $(\cdot)^*$ is the dual.

- equations
  - pentagon equation
  - triangle equation
  - hexagon equations
  - 'yanking' equations for each of $\otimes, I, \multimap$
  - a, b, c, l, r composed with their inverses equal the identity

When we interpret the rewrite

$$c : \otimes'^{*} \tilde{\Rightarrow} \mathbf{curry}^{-1}(\multimap^{*})$$

in **Prof**, we get

$$c : \hom(x \otimes y, z) \tilde{\Rightarrow} \hom(x, y \multimap z) : \mathbf{C}^{\mathrm{op}} \boxtimes C^{\mathrm{op}} \nrightarrow \mathbf{C}^{\mathrm{op}}$$

Start with the tensor

$$\begin{aligned} \otimes\colon \mathbf{C} \times \mathbf{C} &\not\to \mathbf{C} \\ (x, y) &\mapsto z \mapsto \hom(z, x \otimes y), \end{aligned}$$

then prime and take its dual to get

$$\begin{aligned} \otimes'^{*}\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C}^{\mathrm{op}} &\not\to \mathbf{C}^{\mathrm{op}} \\ (x, y) &\mapsto z \mapsto \hom(x \otimes y, z), \end{aligned}$$

which is the first half of the currying isomorphism.

Now start with the internal hom:

$$\begin{aligned} \multimap : \mathbf{C}^{op} \times \mathbf{C} &\nrightarrow \mathbf{C} \\ (y, z) &\mapsto x \mapsto \hom(x, y \multimap z), \end{aligned}$$

Take the dual:

$$\begin{aligned} \multimap^* : \mathbf{C}^{op} &\nrightarrow \mathbf{C} \times \mathbf{C}^{\mathrm{op}} \\ x &\mapsto (y, z) \mapsto \hom(x, y \multimap z), \end{aligned}$$

Uncurry:

$$\begin{aligned} \mathbf{curry}^{-1}(\multimap^*) : \mathbf{C}^{\mathrm{op}} \times \mathbf{C}^{\mathrm{op}} &\nrightarrow \mathbf{C}^{\mathrm{op}} \\ (x, y) &\mapsto z \mapsto \hom(x, y \multimap z), \end{aligned}$$

which is the second half of the currying isomorphism.

Just as monoidal categories are the right place to talk about monoids, compact closed bicategories are the right place to talk about monoidal closed categories, *e.g.* **programming languages**.