WEIGHTED COMMUTATORS IN SEMI-ABELIAN CATEGORIES

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OUTLINE

1. INTRODUCTION (HUQ CENTRALITY, SMITH CENTRALITY)

2. WEIGHTED CENTRALITY (MULTIPLICATIVE WEIGHTED COSPANS)

3. NORMAL WEIGHTED COMMUTATOR (DEFINITION, MAIN RESULT, EXAMPLE)

1. INTRODUCTION

"CENTRALITY" OF ARROWS: $\mathcal{C}$ — POINTED, BINARY PRODUCTS

GIVEN ARROWS WITH THE SAME CODOMAIN

$$X \xrightarrow{x} A \xleftarrow{y} Y$$

THEY **COMMUTE** IF THERE IS A $\varphi : X \times Y \longrightarrow A$

SUCH THAT

$$\begin{tikzcd} & X\times Y \arrow[d, "\varphi"] & \\ X \arrow[ur, "\langle 1,0\rangle"] \arrow[r, "x"'] & A & Y \arrow[ul, "\langle 0,1\rangle"'] \arrow[l, "y"] \end{tikzcd}$$

$$\varphi \cdot \langle 1,0\rangle = x \qquad \text{AND} \qquad \varphi \cdot \langle 0,1\rangle = y$$

IN THIS CASE ONE WRITES $[X, Y]_{\text{Huq}} = 0$

EXAMPLES

$\mathcal{C}$ = GRP LET $X \rightarrowtail^{x} A$ AND $Y \rightarrowtail^{y} A$

BE INCLUSIONS OF NORMAL SUBGROUPS

$$[X,Y]_{HUQ} = 0 \iff [X,Y] = \{1\}$$

$\mathcal{C}$ = RNG LET $I \rightarrowtail^{i} A$ AND $J \rightarrowtail^{j} A$

BE INCLUSIONS OF IDEALS

$$[I,J]_{HUQ} = 0 \iff I \cdot J + J \cdot I = (0)$$

HUQ'S WORK (1968) CONCERNS THE CENTRALITY OF ARROWS.

A DIFFERENT NOTION OF CENTRALITY WAS INTRODUCED BY SMITH (1976): CENTRALITY OF EQUIVALENCE RELATIONS

"CENTRALITY" OF EQUIVALENCE RELATIONS

$\mathcal{C}$ — MAL'TSEV CATEGORY
— WITH PULLBACKS

GIVEN EQUIVALENCE RELATIONS $R \underset{p_2}{\overset{p_1}{\rightrightarrows}} A$ AND $S \underset{p_2}{\overset{p_1}{\rightrightarrows}} A$ ON THE SAME OBJECT A, ONE FORMS

$$\begin{tikzcd} & R \times_A S \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ R \arrow[dr, "p_2"'] & & S \arrow[dl, "p_1"] \\ & A & \end{tikzcd}$$

THEN: R AND S **CENTRALISE** EACH OTHER

IF THERE IS A $p : R \times_A S \longrightarrow A$ SUCH THAT

$$\begin{tikzcd} & R \times_A S \arrow[d, dashed, "p"] & \\ R \arrow[ur, "\langle 1, \Delta \cdot p_2 \rangle"] \arrow[r, "p_1"'] & A & S \arrow[ul, "\langle \Delta \cdot p_1, 1 \rangle"'] \arrow[l, "p_2"] \end{tikzcd}$$

$$p \cdot \langle 1, \Delta \cdot p_2 \rangle = p_1$$
$$p \cdot \langle \Delta \cdot p_1, 1 \rangle = p_2$$

"INTERNALLY": $p(x,y,y) = x$ AND $p(x,x,y) = y$

SO THAT p IS A PARTIAL MAL'TSEV OPERATION.

IN 1995 PEDICCHIO OBSERVED THAT:

THE EXISTENCE OF $p : R \times_A S \to A$

WITH $\begin{cases} p(x,y,y) = x \\ p(x,x,y) = y \end{cases}$

$$\Updownarrow$$

$$[R,S]_{SMITH} = \Delta_A$$

IN 2002 WITH BOURN WE INVESTIGATED THE RELATIONSHIP BETWEEN THE TWO NOTIONS OF CENTRALITY.

$\mathcal{C}$ POINTED MAL'TSEV

WITH ANY EQUIVALENCE RELATION $R \underset{p_2}{\overset{p_1}{\rightrightarrows}} A$ ONE CAN ASSOCIATE THE CORRESPONDING NORMAL MONOMORPHISM $I_R \rightarrowtail A$.

WE OBSERVED THAT:

$$[R,S]_{SMITH} = \Delta_A \Rightarrow [I_R, I_S]_{HUQ} = 0$$

THE CONVERSE IMPLICATION

$$[R,S]_{\text{SMITH}} = \Delta_A \quad \Longleftarrow \quad [I_R, I_S]_{\text{HUQ}} = 0$$

HOLDS IN MANY CATEGORIES:

- STRONGLY PROTOMODULAR (BOURN-GRAN, 2002)
- ACTION ACCESSIBLE (BOURN-JANELIDZE, 2009)
- ALGEBRAIC EXPONENTIATION (GRAY, 2010)

BUT NOT IN GENERAL: COUNTER-EXAMPLES IN THE **SEMI-ABELIAN CATEGORY** OF **DIGROUPS**, DUE TO JANELIDZE (2004), AND ... IN THE NEXT TALK BY TIM VAN DER LINDEN!

NATURAL QUESTION:

IS THERE A GENERAL NOTION OF CENTRALITY THAT GIVES BOTH **HUQ CENTRALITY** AND **SMITH CENTRALITY** AS SPECIAL CASES?

2. WEIGHTED CENTRALITY

FROM NOW ON $\mathcal{C}$ WILL DENOTE A **SEMI-ABELIAN CATEGORY** (JANELIDZE, MÁRKI, THOLEN - 2002):

- $\mathcal{C}$ FINITELY COMPLETE
- EXACT
- POINTED, 0
- PROTOMODULAR:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "u"'] & A \arrow[r, shift left] \arrow[d, "v"'] & B \arrow[r] \arrow[l, shift left] \arrow[d, "w"] & 0 \\
0 \arrow[r] & K' \arrow[r] & A' \arrow[r, shift left] & B' \arrow[r] \arrow[l, shift left] & 0
\end{tikzcd}$$

u AND w ISOS $\Rightarrow$ v ISO

- BINARY COPRODUCTS

EXAMPLES

GRP, RNG, Ω-GRP, GRP(COMP), GRP(PROF), SET_*^{op}, HEYT, ETC.

WEIGHTED COSPAN

$$\begin{tikzcd} & W \arrow[d, "w"] & \\ X \arrow[r, "x"'] & A & Y \arrow[l, "y"] \end{tikzcd}$$

ONE FORMS THE PULLBACK

$$\begin{tikzcd} & (W+X)\times_W (W+Y) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ W+X \arrow[dr, "{[1,0]}"'] & & W+Y \arrow[dl, "{[1,0]}"] \\ & W & \end{tikzcd}$$

THAT INDUCES THE "INCLUSIONS"

$$\begin{tikzcd} W+X \arrow[r, dashed, "{\langle 1, i_1\cdot[1,0]\rangle}"] & (W+X)\times_W (W+Y) & W+Y \arrow[l, dashed, "{\langle i_1\cdot[1,0], 1\rangle}"'] \end{tikzcd},$$

DEFINITION

AN **INTERNAL MULTIPLICATION** $X \times Y \to A$ **OVER** $w : W \longrightarrow A$ IS AN ARROW m

$$
\begin{tikzcd}
W+X \arrow[r, "{\langle 1,\, i_1 \cdot [1,0] \rangle}"] \arrow[dr, "{[w,x]}"'] & (W+X) \times_W (W+Y) \arrow[d, dashed, "m"] & W+Y \arrow[l, "{\langle i_1 \cdot [1,0],\, 1 \rangle}"'] \arrow[dl, "{[w,y]}"] \\
& A &
\end{tikzcd}
$$

MAKING THE TRIANGLES COMMUTE.

NOTATION

$$[(X,x),(Y,y)]_{(W,w)} = 0 \text{ , OR}$$

$$[x,y]_w = 0 \quad \text{FOR SHORT.}$$

REMARKS

a) m IS **UNIQUE**, WHEN IT EXISTS.

b) IF $W \xrightarrow{w} A = 0 \xrightarrow{0} A$, ONE GETS

$$
\begin{tikzcd}
X \arrow[r, "{\langle 1,0 \rangle}"] \arrow[dr, "x"'] & X \times Y \arrow[d, dashed, "m"] & Y \arrow[l, "{\langle 0,1 \rangle}"'] \arrow[dl, "y"] \\
& A &
\end{tikzcd}
$$

$\Rightarrow$ **HUQ CENTRALITY!**

WHEN $\mathcal{C}$ IS **SEMI-ABELIAN**,

NORMAL MONOS $\longleftrightarrow$ EFFECTIVE EQUIVALENCE RELATIONS

GIVEN A **WEIGHTED COSPAN**

$$\begin{tikzcd} & W \arrow[d, "w"] & \\ X \arrow[r, tail, "x"'] & A & Y \arrow[l, tail, "y"] \end{tikzcd}$$

WITH x AND y NORMAL MONOS, ONE GETS A SPAN

$$\begin{tikzcd} & A \arrow[dl, two heads, "q_X"'] \arrow[dr, two heads, "q_Y"] & \\ A/X & & A/Y \end{tikzcd}$$

WHOSE KERNEL PAIRS ARE THE EFFECTIVE EQUIVALENCE RELATIONS R_X AND R_Y:

$$\begin{tikzcd} R_X \arrow[dr, shift left] \arrow[dr, shift right] & & R_Y \arrow[dl, shift left] \arrow[dl, shift right] \\ & A \arrow[dl, two heads, "q_X"'] \arrow[dr, two heads, "q_Y"] & \\ A/X & & A/Y \end{tikzcd}$$

A PARTIAL MAL'TSEV OPERATION $p : R_X \times_A R_Y \to A$

ON THE EQUIVALENCE RELATIONS R_X AND R_Y

$$\begin{tikzcd}
 & R_X \times_A R_Y \arrow[dl] \arrow[dr] \arrow[dd, "p"] & \\
R_X \arrow[dr, "p_2"'] & & R_Y \arrow[dl, "p_1"] \\
 & A &
\end{tikzcd}$$

ALWAYS INDUCES A **MULTIPLICATION**

$X \times Y \to A$ OVER $W : W \to A$:

$$\begin{tikzcd}
 & & R_X \times_A R_Y \arrow[dl] \arrow[dr] \arrow[dd, "p"] & & \\
 & R_X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & R_Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A \arrow[dr, "q_X"'] & & A \arrow[dl, "q_X"] \arrow[dr, "q_Y"'] & & A \arrow[dl, "q_Y"] \\
 & A/X & & A/Y &
\end{tikzcd}$$

A PARTIAL MAL'TSEV OPERATION $p : R_X \times_A R_Y \to A$

ON THE EQUIVALENCE RELATIONS R_X AND R_Y

$$
\begin{tikzcd}
 & R_X \times_A R_Y \arrow[dl] \arrow[dr] \arrow[dd, "p"] & \\
R_X \arrow[dr, "p_2"'] & & R_Y \arrow[dl, "p_1"] \\
 & A &
\end{tikzcd}
$$

ALWAYS INDUCES A **MULTIPLICATION** $p \cdot \gamma_W$

$X \times Y \longrightarrow A$ **OVER** $w : W \longrightarrow A$:

$$
\begin{tikzcd}
 & & & (W+X) \times_W (W+Y) \arrow[dll, "\pi_1"'] \arrow[drr, "\pi_2"] \arrow[d, dashed, "\gamma_W"] & & & \\
 & W+X \arrow[dd, "{[w,x]}"'] \arrow[ddrr, "{[w,0]}"] \arrow[dr, dashed, "\alpha"] & & R_X \times_A R_Y \arrow[dl] \arrow[dr] \arrow[dd, "p"] & & W+Y \arrow[ddll, "{[w,0]}"'] \arrow[dd, "{[w,y]}"] \arrow[dl, dashed, "\beta"'] & \\
 & & R_X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & R_Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
 & A \arrow[dr, "q_X"'] & & A \arrow[dl, "q_X"] \arrow[dr, "q_Y"'] & & A \arrow[dl, "q_Y"] & \\
 & & A/X & & A/Y & &
\end{tikzcd}
$$

THIS SHOWS THAT

$$[R_x, R_y]_{\text{SMITH}} = \Delta_A \Rightarrow [x, y]_{(w,w)} = 0$$

<u>THEOREM</u> $\mathcal{C}$ SEMI-ABELIAN, $X \overset{x}{\rightarrowtail} A$ AND $Y \overset{y}{\rightarrowtail} A$ NORMAL MONOS. THEN:

$$[R_x, R_y]_{\text{SMITH}} = \Delta_A \Leftrightarrow [x, y]_{(A, 1_A)} = 0$$

3. NORMAL WEIGHTED COMMUTATOR

THE WEIGHTED COSPANS IN $\mathcal{C}$

$$\begin{tikzcd} & W \arrow[d, "w"'] & \\ X \arrow[r, "x"'] & A & Y \arrow[l, "y"] \end{tikzcd}$$

FORM A CATEGORY, DENOTED BY $CS_W(\mathcal{C})$.

A WEIGHTED COSPAN IN $\mathcal{C}$ IS MULTIPLICATIVE WHEN THERE IS A MULTIPLICATION $X \times Y \to A$ OVER $w : W \to A$. THERE IS A FUNCTOR

$$U : MCS_W(\mathcal{C}) \longrightarrow CS_W(\mathcal{C})$$

THAT "FORGETS" THE MULTIPLICATION.

IS THERE A LEFT ADJOINT

$$\begin{tikzcd} MCS_{\mathbb{W}}(\mathcal{C}) \arrow[r, "U"'] \arrow[r, phantom, "\perp"] & CS_{\mathbb{W}}(\mathcal{C}) \arrow[l, bend right, dashed, "F"'] \end{tikzcd} \quad ?$$

YES!

ON OBJECTS, THE FUNCTOR $F : CS_{\mathbb{W}}(\mathcal{C}) \to MCS_{\mathbb{W}}(\mathcal{C})$ IS DEFINED AS FOLLOWS:

GIVEN

$$C = \left(\begin{tikzcd} & \mathbb{W} \arrow[d, "w"] & \\ X \arrow[r, "x"'] & A & Y \arrow[l, "y"] \end{tikzcd} \right) \in CS_{\mathbb{W}}(\mathcal{C})$$

ONE FORMS THE PUSHOUT

$$\begin{tikzcd} \mathbb{W}+X+Y \arrow[r, "{[w,x,y]}"] \arrow[d, "{\langle [i_1,i_2,0],[i_1,0,i_2] \rangle}"'] & A \arrow[d, "\eta_c"] \\ (\mathbb{W}+X) \times_{\mathbb{W}} (\mathbb{W}+Y) \arrow[r] & \tilde{C} \end{tikzcd}$$

THEN:

$$F(C) = \left(\begin{tikzcd} & \mathbb{W} \arrow[d, "\eta_c \cdot w"] & \\ X \arrow[r, "\eta_c \cdot x"'] & \tilde{C} & Y \arrow[l, "\eta_c \cdot y"] \end{tikzcd} \right)$$

DEFINITION

THE (W,w)-**WEIGHTED NORMAL COMMUTATOR**

$N[x,y]_{(W,\, W \xrightarrow{w} A)}$ OF X AND Y IS THE KERNEL

OF $\eta_c : A \longrightarrow \tilde{C}$:

$$0 \longrightarrow N[x,y]_{(W,\, W \xrightarrow{w} A)} \rightarrowtail A \xrightarrow{\eta_c} \tilde{C} \longrightarrow 0$$

THEOREM

$\mathcal{C}$ SEMI-ABELIAN.

GIVEN $X \xrightarrow{x} A \xleftarrow{y} Y$, ONE HAS :

1) $N[x,y]_{(0,\, 0 \xrightarrow{0} A)} = [x,y]_{HUQ}$

2) IF, MOREOVER, $X \overset{x}{\rightarrowtail} A \overset{y}{\leftarrowtail} Y$ ARE NORMAL MONOS, ONE HAS :

$$N[x,y]_{(A,\, A \xrightarrow{1} A)} = {}^{0}\!\big/_{[R_x,\, R_y]_{SMITH}}$$

EXAMPLE URSINI'S COMMUTATOR

LET $\mathbb{C}$ BE A SEMI-ABELIAN VARIETY OF UNIVERSAL ALGEBRAS.

URSINI IN 1994 INTRODUCED THE COMMUTATOR TERMS AS FOLLOWS:

A TERM $t(w_1, \ldots, w_k, x_1, \ldots, x_m, y_1, \ldots, y_m)$ IS A COMMUTATOR TERM IN $\vec{x} = (x_1, \ldots, x_m)$ AND $\vec{y} = (y_1, \ldots, y_m)$ IF

1. $t(w_1, \ldots, w_k, 0, \ldots, 0, y_1, \ldots, y_m) = 0$

AND

2. $t(w_1, \ldots, w_k, x_1, \ldots, x_m, 0, \ldots, 0) = 0$

WE WRITE $CT_{\mathbb{C}}(\vec{x}, \vec{y})$ FOR THE COLLECTION OF ALL SUCH COMMUTATOR TERMS.

DEFINITION

GIVEN NORMAL SUBALGEBRAS $X \rightarrowtail A \leftarrowtail Y$,

$$[X,Y]_{\text{URSINI}} = \{ t_A(w_1,\dots,w_k,x_1,\dots,x_m,y_1,\dots,y_m) \mid$$
$$w_1,\dots,w_k \in A;\ x_1,\dots,x_m \in X;\ y_1,\dots,y_m \in Y;$$
$$t \in CT_{\mathbb{C}}(\vec{x},\vec{y}) \}$$

THEOREM

LET $\mathbb{C}$ BE A SEMI-ABELIAN VARIETY.

THEN

$$[X,Y]_{\text{URSINI}} = N[X,Y]_{(A,\,A \xrightarrow{1} A)}$$

Remark

It is possible to define, more generally, a (W, w)-**weighted commutator**

$$[X, Y]_{(W,\, W \xrightarrow{w} A)}$$

of X and Y.

In particular, the case where

$W \xrightarrow{w} A = 0 \xrightarrow{0} A$ yields **Higgins' commutator**: $[X, Y]_{(0,\, 0 \to A)} = [X, Y]_{\text{Higgins}}$

It turns out that:

$$\overline{[X, Y]_{(W,\, W \xrightarrow{w} A)}}^{\text{normal}} = N[X, Y]_{(W,\, W \xrightarrow{w} A)}$$

(See also Mantovani - Metere, 2010)

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