

WEIGHTED COMMUTATORS  
IN SEMI-ABELIAN CATEGORIES

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# WORK IN COLLABORATION WITH GEORGE JANELIDZE AND ALDO URSINI

## OUTLINE

1. INTRODUCTION (HUG CENTRALITY,  
SMITH CENTRALITY)

2. WEIGHTED CENTRALITY  
(MULTIPLICATIVE WEIGHTED COSPANS)

3. NORMAL WEIGHTED COMMUTATOR  
(DEFINITION, MAIN RESULT, EXAMPLE)

# I. INTRODUCTION

"CENTRALITY" OF ARROWS:

$\mathcal{C}$  - POINTED  
- BINARY PRODUCTS

GIVEN ARROWS WITH THE SAME CODOMAIN

$$X \xrightarrow{x} A \xleftarrow{y} Y$$

THEY **COMMUTE** IF THERE IS A  $\varphi: X \times Y \rightarrow A$   
SUCH THAT

$$\begin{array}{ccc} & X \times Y & \\ \langle 1, 0 \rangle \nearrow & \downarrow \varphi & \nwarrow \langle 0, 1 \rangle \\ X & \xrightarrow{x} A \xleftarrow{y} & Y \end{array}$$

$$\varphi \cdot \langle 1, 0 \rangle = x \quad \text{AND} \quad \varphi \cdot \langle 0, 1 \rangle = y$$

IN THIS CASE ONE WRITES  $[X, Y] = 0$   
Hvz

## EXAMPLES

$\mathcal{C} = \text{GRP}$  LET  $X \xrightarrow{x} A$  AND  $Y \xrightarrow{y} A$

BE INCLUSIONS OF **NORMAL SUBGROUPS**

$$[X, Y]_{\text{HUQ}} = 0 \quad (\Leftrightarrow) \quad [x, y] = \{1\}$$

$\mathcal{C} = \text{RNG}$  LET  $I \xrightarrow{i} A$  AND  $J \xrightarrow{j} A$

BE INCLUSIONS OF **IDEALS**

$$[I, J]_{\text{HUQ}} = 0 \quad (\Leftrightarrow) \quad I \cdot J + J \cdot I = (0)$$

HUQ'S WORK (1968) CONCERNS THE **CENTRALITY OF ARROWS**.

A DIFFERENT NOTION OF CENTRALITY WAS INTRODUCED BY SMITH (1976):

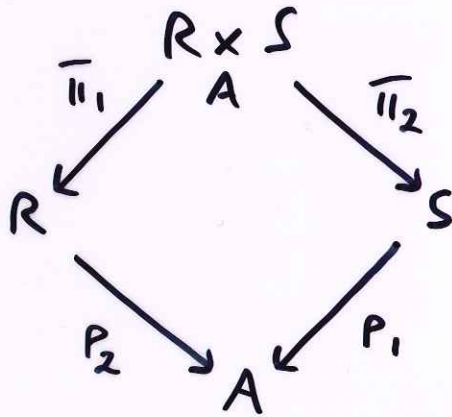
**CENTRALITY OF EQUIVALENCE RELATIONS**



# "CENTRALITY" OF EQUIVALENCE RELATIONS

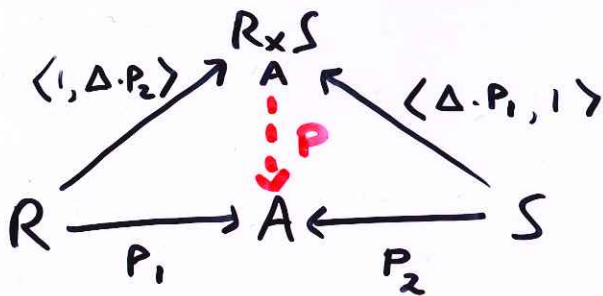
$\mathcal{C}$  MAL'TSEV CATEGORY  
WITH PULLBACKS

GIVEN EQUIVALENCE RELATIONS  $R \begin{matrix} \xrightarrow{P_1} \\ \xleftarrow{P_2} \end{matrix} A$  AND  $S \begin{matrix} \xrightarrow{P_1} \\ \xleftarrow{P_2} \end{matrix} A$  ON THE SAME OBJECT  $A$ , ONE FORMS



THEN:  $R$  AND  $S$  **CENTRALISE** EACH OTHER

IF THERE IS A  $P: R \times_S A \longrightarrow A$  SUCH THAT



$$P \cdot \langle 1, \Delta \cdot P_2 \rangle = P_1$$

$$P \cdot \langle \Delta \cdot P_1, 1 \rangle = P_2$$

"INTERNALLY":  $P(x, y, y) = x$  AND  $P(x, x, y) = y$

SO THAT  $P$  IS A PARTIAL MAL'TSEV OPERATION.

IN 1995 PEDICCHIO OBSERVED THAT :

THE EXISTENCE OF  $P: R \times_S A \rightarrow A$

WITH  $\begin{cases} P(x, y, y) = x \\ P(x, x, y) = y \end{cases}$

$\hat{=}$

$[R, S]_{\text{SMITH}} = \Delta_A$

IN 2002 WITH BOURN WE INVESTIGATED  
THE RELATIONSHIP BETWEEN THE TWO NOTIONS  
OF CENTRALITY.

$\mathcal{C}$  POINTED MAL'TSEV

WITH ANY EQUIVALENCE RELATION  $R \begin{matrix} \xrightarrow{P_1} \\ \xleftrightarrow{\quad} \\ \xleftarrow{P_2} \end{matrix} A$

ONE CAN ASSOCIATE THE CORRESPONDING

NORMAL HOMOMORPHISM  $I_R \twoheadrightarrow A$ .

WE OBSERVED THAT :

$[R, S]_{\text{SMITH}} = \Delta_A \Rightarrow [I_R, I_S]_{\text{HUR}} = 0$

## THE CONVERSE IMPLICATION

$$[R, S]_{\text{SMITH}} = \Delta_A \iff [I_R, I_S]_{\text{HUQ}} = 0$$

HOLDS IN MANY CATEGORIES:

- STRONGLY PROTOMODULAR (BOURN-GRAY, 2002)
- ACTION ACCESSIBLE (BOURN-JAMELIDZE, 2009)
- ALGEBRAIC EXPONENTIATION (GRAY, 2010)

BUT NOT IN GENERAL: COUNTER-EXAMPLES IN THE **SEMI-ABELIAN CATEGORY OF DIGROUPS**, DUE TO JAMELIDZE (2004), AND ... IN THE NEXT TALK BY TIM VAN DER LINDEN!

## NATURAL QUESTION:

IS THERE A GENERAL NOTION OF CENTRALITY THAT GIVES BOTH **HUQ CENTRALITY** AND **SMITH CENTRALITY** AS SPECIAL CASES?



## 2. WEIGHTED CENTRALITY

FROM NOW ON  $\mathcal{C}$  WILL DENOTE A  
**SEMI-ABELIAN CATEGORY** (JANELIDZE,  
MÁRKI, THOLEN - 2002):

- $\mathcal{C}$  FINITELY COMPLETE
- EXACT
- POINTED,  $0$
- PROTOMODULAR:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & A & \rightleftarrows & B & \longrightarrow & 0 \\ & & u \downarrow & & v \downarrow & & \downarrow w & & \\ 0 & \longrightarrow & K' & \longrightarrow & A' & \rightleftarrows & B' & \longrightarrow & 0 \end{array}$$

$$u \text{ AND } w \text{ ISOS} \Rightarrow v \text{ ISO}$$

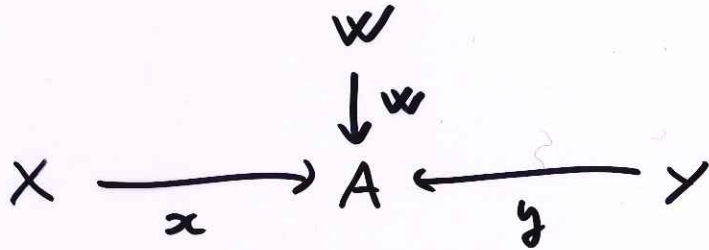
- BINARY COPRODUCTS

### EXAMPLES

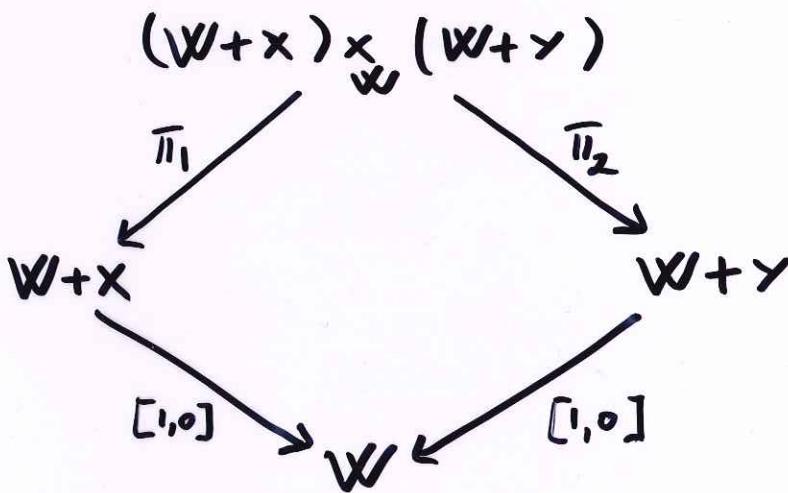
GRP, RMG,  $\Omega$ -GRP, GRP(COMP), GRP(PROF),  
 $\text{SET}_*^{\text{OP}}$ , HEYT, ETC.



# WEIGHTED COSPAN



ONE FORMS THE PULLBACK

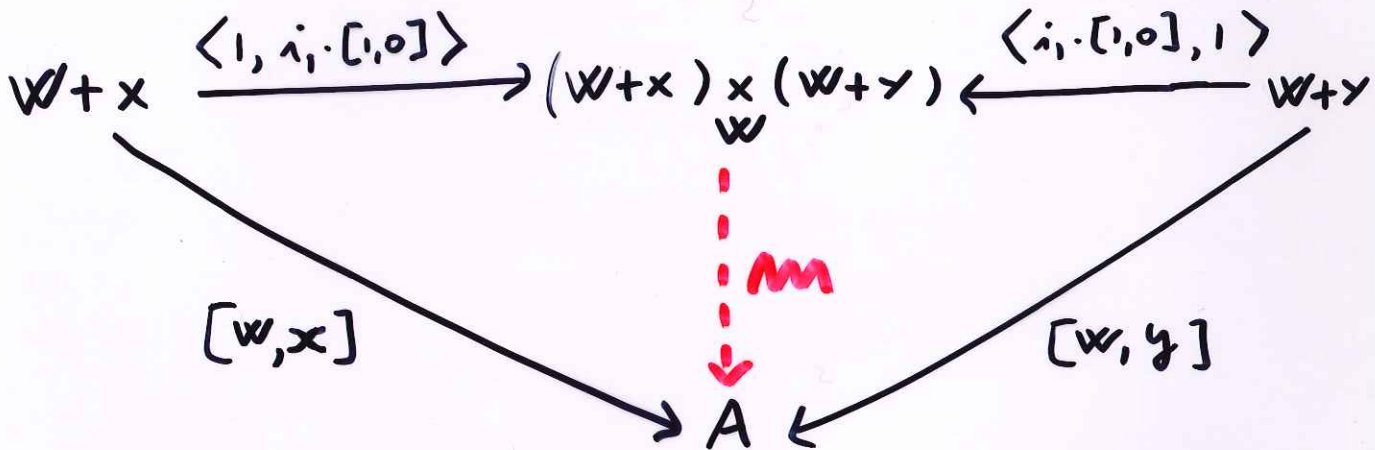


THAT INDUCES THE "INCLUSIONS"

$$\begin{array}{c}
 \langle 1, \lambda_i \cdot [1,0] \rangle \qquad \qquad \qquad \langle \lambda_i \cdot [1,0], 1 \rangle \\
 W+x \text{ --- } \rightarrow (W+x) \times_{\substack{W \\ w}} (W+y) \leftarrow \text{ --- } W+y
 \end{array}$$

# DEFINITION

AN **INTERNAL MULTIPLICATION**  $x \times y \rightarrow A$   
**OVER**  $w: W \rightarrow A$  IS AN ARROW  $m$



MAKING THE TRIANGLES COMMUTE.

## NOTATION

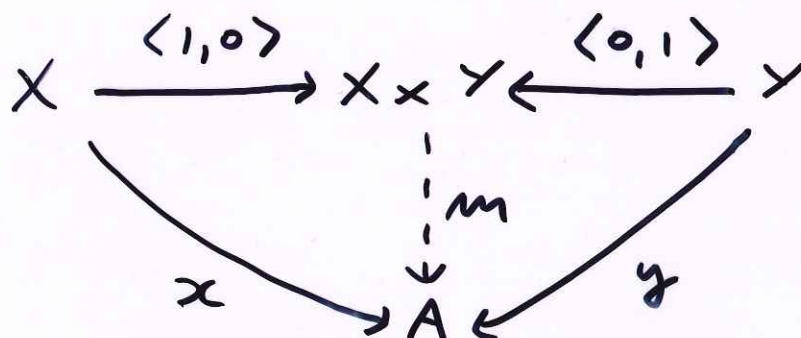
$$[(x, x), (y, y)]_{(w, w)} = 0, \text{ OR}$$

$$[x, y]_w = 0 \text{ FOR SHORT.}$$

## REMARKS

a)  $m$  IS **UNIQUE**, WHEN IT EXISTS.

b) IF  $w \xrightarrow{w} A = 0 \xrightarrow{0} A$ , ONE GETS

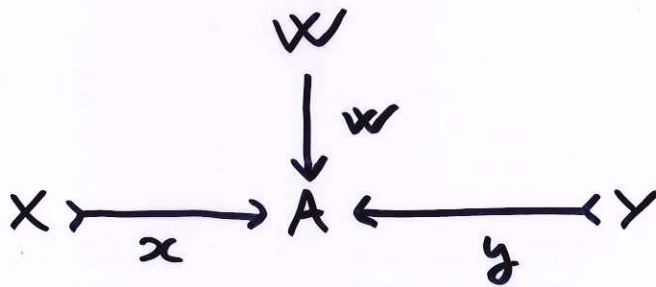


$\Rightarrow$  **HUQ CENTRALITY!**

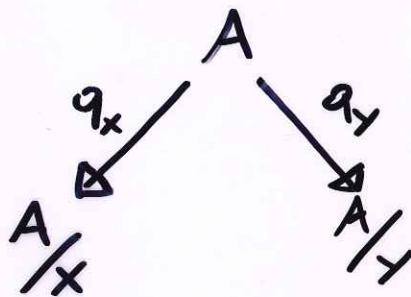
WHEN  $\mathcal{C}$  IS **SEMI-ABELIAN**,

NORMAL MONOS  $\longleftrightarrow$  EFFECTIVE EQUIVALENCE RELATIONS

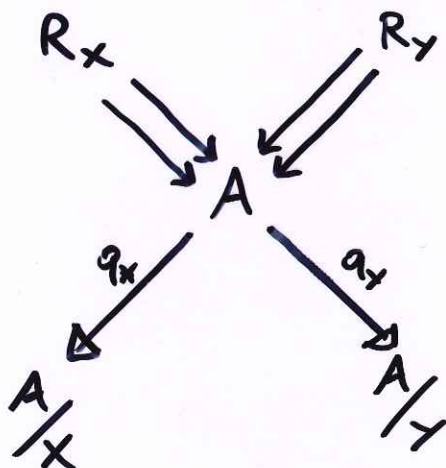
GIVEN A **WEIGHTED COSPAN**



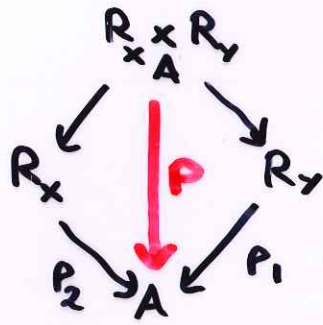
WITH  $x$  AND  $y$  NORMAL MONOS, ONE GETS A SPAN



WHOSE KERNEL PAIRS ARE THE EFFECTIVE EQUIVALENCE RELATIONS  $R_x$  AND  $R_y$ :

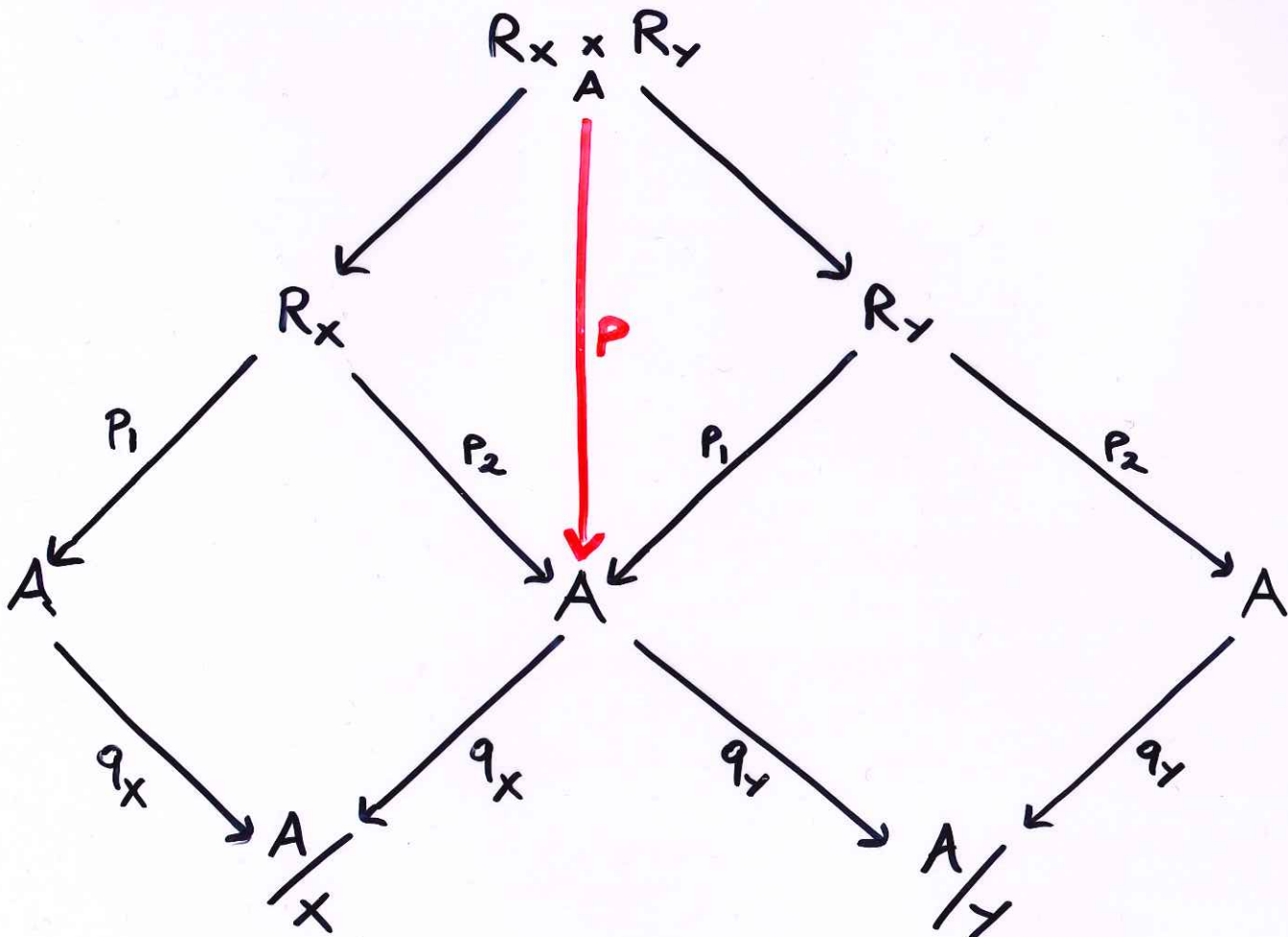


A PARTIAL MAL'TSEV OPERATION  $P: R_x \times_A R_y \rightarrow A$   
 ON THE EQUIVALENCE RELATIONS  $R_x$  AND  $R_y$



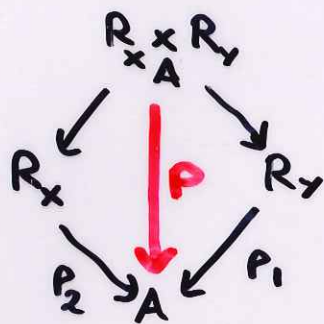
ALWAYS INDUCES A MULTIPLICATION

$X \times Y \rightarrow A$  OVER  $W: W \rightarrow A$  :

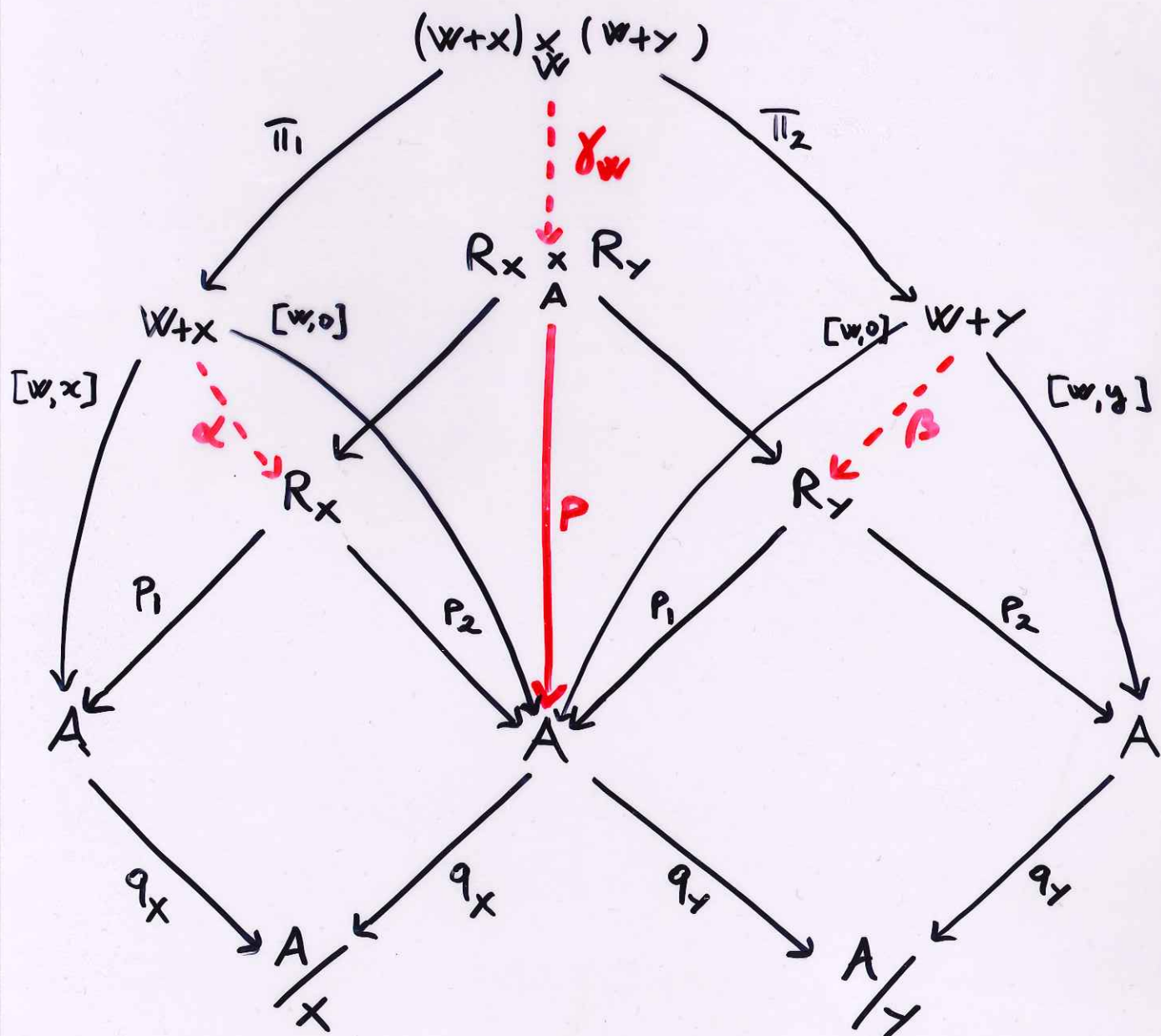




A PARTIAL MAL'TSEV OPERATION  $P: R_x \times_A R_y \rightarrow A$   
 ON THE EQUIVALENCE RELATIONS  $R_x$  AND  $R_y$



ALWAYS INDUCES A MULTIPLICATION  $P \cdot \gamma_w$   
 $X \times Y \rightarrow A$  OVER  $W: W \rightarrow A$ :



THIS SHOWS THAT

$$[R_x, R_y]_{\text{SMITH}} = \Delta_A \Rightarrow [x, y]_{(w, w)} = 0$$

THEOREM  $\mathcal{C}$  SEMI-ABELIAN,  $X \xrightarrow{x} A$  AND  $Y \xrightarrow{y} A$  NORMAL MONOS. THEN:

$$[R_x, R_y]_{\text{SMITH}} = \Delta_A \Leftrightarrow [x, y]_{(A, A)} = 0$$

### 3. NORMAL WEIGHTED COMMUTATOR

THE WEIGHTED COSPANS IN  $\mathcal{C}$

$$\begin{array}{ccc} & w & \\ & \downarrow & \\ X & \xrightarrow{x} & A \xleftarrow{y} Y \\ & & w \end{array}$$

FORM A CATEGORY, DENOTED BY  $CS_w(\mathcal{C})$ .

A WEIGHTED COSPAN IN  $\mathcal{C}$  IS **MULTIPLICATIVE** WHEN THERE IS A MULTIPLICATION  $X \times Y \rightarrow A$  OVER  $w: W \rightarrow A$ . THERE IS A FUNCTOR

$$U: MCS_w(\mathcal{C}) \longrightarrow CS_w(\mathcal{C})$$

THAT "FORGETS" THE MULTIPLICATION.

IS THERE A LEFT ADJOINT

$$\text{MCS}_w(\varphi) \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{U} \end{array} \text{CS}_w(\varphi) \quad ?$$

YES!

ON OBJECTS, THE FUNCTOR  $F: \text{CS}_w(\varphi) \rightarrow \text{MCS}_w(\varphi)$

IS DEFINED AS FOLLOWS:

GIVEN

$$C = \left( \begin{array}{ccc} & w & \\ & \downarrow w & \\ x & \xrightarrow{x} & A \leftarrow y & y \end{array} \right) \in \text{CS}_w(\varphi)$$

ONE FORMS THE PUSHOUT

$$\begin{array}{ccc} w+x+y & \xrightarrow{[w,x,y]} & A \\ \downarrow \langle [i_1, i_2, 0], [i_1, 0, i_2] \rangle & & \downarrow \eta_c \\ (w+x) \underset{w}{\times} (w+y) & \longrightarrow & \tilde{c} \end{array}$$

THEN:

$$F(C) = \left( \begin{array}{ccc} & w & \\ & \downarrow \eta_c \cdot w & \\ x & \xrightarrow{\eta_c \cdot x} & \tilde{c} \leftarrow \eta_c \cdot y & y \end{array} \right)$$



## DEFINITION

THE  $(W, w)$ -WEIGHTED NORMAL COMMUTATOR

$N[x, y]_{(W, w \rightarrow A)}$  OF  $X$  AND  $Y$  IS THE KERNEL

OF  $\eta_c: A \rightarrow \tilde{C}$ :

$$0 \longrightarrow N[x, y]_{(W, w \rightarrow A)} \longrightarrow A \xrightarrow{\eta_c} \tilde{C} \longrightarrow 0$$

## THEOREM $\mathcal{C}$ SEMI-ABELIAN.

GIVEN  $X \xrightarrow{x} A \xleftarrow{y} Y$ , ONE HAS:

$$1) N[x, y]_{(0, 0 \rightarrow A)} = [x, y]_{\text{HUQ}}$$

2) IF, MOREOVER,  $X \xrightarrow{x} A \xleftarrow{y} Y$  ARE NORMAL MONOS, ONE HAS:

$$N[x, y]_{(A, A \rightarrow A)} = \frac{0}{[R_x, R_y]_{\text{SMITH}}}$$



## EXAMPLE URSINI'S COMMUTATOR

LET  $\mathcal{C}$  BE A SEMI-ABELIAN VARIETY OF UNIVERSAL ALGEBRAS.

URSINI IN 1994 INTRODUCED THE

COMMUTATOR TERMS AS FOLLOWS:

A TERM  $t(w_1, \dots, w_k, x_1, \dots, x_m, y_1, \dots, y_m)$

IS A COMMUTATOR TERM IN  $\vec{x} = (x_1, \dots, x_m)$  AND

$\vec{y} = (y_1, \dots, y_m)$  IF

1.  $t(w_1, \dots, w_k, 0, \dots, 0, y_1, \dots, y_m) = 0$

AND

2.  $t(w_1, \dots, w_k, x_1, \dots, x_m, 0, \dots, 0) = 0$

WE WRITE  $CT_{\mathcal{C}}(\vec{x}, \vec{y})$  FOR THE

COLLECTION OF ALL SUCH COMMUTATOR TERMS.

## DEFINITION

GIVEN NORMAL SUBALGEBRAS  $X \twoheadrightarrow A \leftarrow Y$ ,

$$[X, Y]_{\text{URSINI}} = \left\{ t_A(w_1, \dots, w_k, x_1, \dots, x_m, y_1, \dots, y_m) \mid \right. \\ \left. w_1, \dots, w_k \in A; x_1, \dots, x_m \in X; y_1, \dots, y_m \in Y; \right. \\ \left. t \in (T_{\mathbb{C}}(\vec{x}, \vec{y})) \right\}$$

## THEOREM

LET  $\mathbb{C}$  BE A SEMI-ABELIAN VARIETY.

THEN

$$[X, Y]_{\text{URSINI}} = N[X, Y]_{(A, A \xrightarrow{1} A)}$$

## REMARK

IT IS POSSIBLE TO DEFINE, MORE GENERALLY,

A  $(W, W)$ -WEIGHTED COMMUTATOR

$[X, Y]_{(W, W \xrightarrow{W} A)}$  OF  $X$  AND  $Y$ .

IN PARTICULAR, THE CASE WHERE

$W \xrightarrow{W} A = 0 \xrightarrow{0} A$  YIELDS HIGGINS'

COMMUTATOR:  $[X, Y]_{(0, 0 \rightarrow A)} = [X, Y]_{\text{HIGGINS}}$

IT TURNS OUT THAT:

$$\overbrace{[X, Y]_{(W, W \xrightarrow{W} A)}}^{\text{NORMAL}} = N [X, Y]_{(W, W \xrightarrow{W} A)}$$

(SEE ALSO MANTOVANI - METERE, 2010)



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