Explanation why a full reflective subcategory $\mathcal{D} \overset{I}{\hookrightarrow} \mathcal{C}$ is closed under all limits which exist in $\mathcal{C}$.

Write $F : \mathcal{C} \to \mathcal{D}$ for the left adjoint to the inclusion I. As I is an inclusion, we can leave it out of the notation (we will include it sometimes to make a point, but usually leave it out). Notice that F being a reflection means $FB \cong B$ for any object B of $\mathcal{D}$ (ie $\varepsilon_B : FIB \to B$ is an iso).

Let $D : \mathcal{J} \to \mathcal{D}$ be a diagram in $\mathcal{D}$, with limit $(L \xrightarrow{\lambda_j} ID(j))$ in $\mathcal{C}$.

Remember that $(A \in \mathcal{C},\ B \in \mathcal{D})$

$$FA \xrightarrow{f} B \quad \longleftrightarrow \quad A \xrightarrow{\eta_A} FA \xrightarrow{If} B$$

$$\text{and} \quad A \xrightarrow{g} B \quad \longleftrightarrow \quad FA \xrightarrow{Fg} FB \xrightarrow[\cong]{\varepsilon_B} B$$

under the adjunction.

So each leg $L \xrightarrow{\lambda_j} D(j)$ corresponds to $FL \xrightarrow{\mu_j} D(j)$ under the adjunction, with $\mu_j \cong F\lambda_j$ and

$$L \xrightarrow{\eta_L} FL \xrightarrow{\mu_j} D(j) = L \xrightarrow{\lambda_j} D(j)$$

(as they both correspond to $FL \xrightarrow{\mu_j} D(j)$).

Now as $\mu_j \cong F\lambda_j$ and for any morphism $D(j) \xrightarrow{D(\alpha)} D(j')$ in the diagram we have $FD(\alpha) \cong D(\alpha)$ (as it lives in $\mathcal{D}$), we see that $FL \xrightarrow{\mu_j} D(j)$ forms a cone over D in $\mathcal{D}$. Moreover, $\eta_L : L \to FL$ is a morphism of cones over ID in $\mathcal{C}$. But L is the limit in $\mathcal{C}$, so we get a unique morphism of cones $c : FL \to L$. The limit property of L immediately gives us $c \circ \eta_L = 1_L$. Now we look at $\eta_L \circ c$.

$$FL \xrightarrow{\eta_L \circ c} FL \iff L \xrightarrow{\eta_L} FL \xrightarrow{c} L \xrightarrow{\eta_L} FL$$
$$= L \xrightarrow{\eta_L} FL$$

But η_L corresponds to the identity of FL, so
$\eta_L \circ c = 1_{FL}$.

Therefore $FL \cong L$, ie L already lies in $\mathcal{D}$.
So $\mathcal{D}$ is closed under all limits which exist in $\mathcal{C}$.