Special Adjoint Functor Theorem

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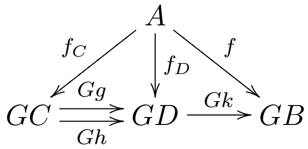
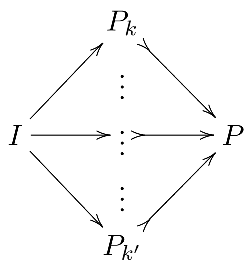
Theorem. *Suppose both $\mathcal{C}$ and $\mathcal{D}$ are locally small, and that $\mathcal{D}$ is complete and well-powered and has a coseparating set. Then a functor $G\colon \mathcal{D} \longrightarrow \mathcal{C}$ has a left adjoint if and only if G preserves small limits.*

Proof. "$\Rightarrow$": G preserves all limits that exist in $\mathcal{D}$ as it is a right adjoint.
"$\Leftarrow$": The "Limits in $(A \downarrow G)$" Lemma implies that each $(A \downarrow G)$ is complete; it also inherits local smallness from $\mathcal{D}$. The Remark "Monos in Functor Categories" implies that the forgetful functor $(A \downarrow G) \longrightarrow \mathcal{D}$ preserves monos (as it creates and so preserves limits by "Limits in $(A \downarrow G)$"), so the subobjects of (B, f) in $(A \downarrow G)$ are those subobjects $B' \rightarrowtail B$ in $\mathcal{D}$ for which $f\colon A \longrightarrow GB$ factors through $GB' \rightarrowtail GB$. So $(A \downarrow G)$ inherits well-poweredness from $\mathcal{D}$.
Given a coseparating set $\mathcal{S}$ for $\mathcal{D}$, the set $\mathcal{S}' = \{(B, f) \mid B \in \mathcal{S},\ f\colon A \longrightarrow GB\}$ (i.e. taking all possible such f) is a coseparating set for $(A \downarrow G)$: if we have $(C, f_C) \overset{g}{\underset{h}{\rightrightarrows}} (D, f_D)$ with $g \neq h$ in $(A \downarrow G)$, there exists $B \in \mathcal{S}$ and $k\colon D \longrightarrow B$ such that $kg \neq kh$. Taking $f = (Gk)f_D$, we have $(B, f) \in \mathcal{S}'$ and $kg \neq kh$ in $(A \downarrow G)$.



Note that $\mathcal{S}'$ really is a set, as $\mathcal{C}$ is locally small.
So we have to show that if a category $\mathcal{A}$ is complete, locally small, well-powered and has a cosparating set, then $\mathcal{A}$ has an initial object I.
Let $\{B_j,\ j \in J\}$ be a coseparating set for $\mathcal{A}$. Form $P = \prod_{j \in J} B_j$ (possible as $\mathcal{A}$ is complete), and a set $\{P_k \rightarrowtail P \mid k \in K\}$ of representatives of subobjects of P (possible as $\mathcal{A}$ is well-powered). Form the limit of the diagram with edges all the $P_k \rightarrowtail P$ for $k \in K$ (possible as $\mathcal{A}$ is complete).

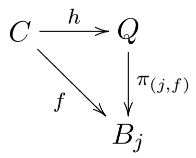


The legs $I \longrightarrow P_k$ are also monos (proof similar to "Pullbacks preserve monos"). We have

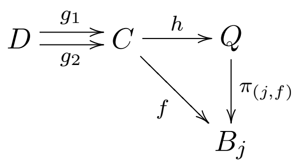
$$(I \rightarrowtail P) \leqslant (P_k \rightarrowtail P)$$

as subobjects, for all $k \in K$. So $I \rightarrowtail P$ is the smallest subobject of P. We want to show that I is initial in $\mathcal{A}$.
First we show that there can be at most one morphism $I \longrightarrow C$ for any $C \in \operatorname{ob} \mathcal{A}$. Suppose we have $I \overset{f}{\underset{g}{\rightrightarrows}} C$. We can form the equaliser $E \rightarrowtail I \overset{f}{\underset{g}{\rightrightarrows}} C$. Then $E \rightarrowtail I \rightarrowtail P$ is a subobject of P, but $I \rightarrowtail P$ is the smallest, so $E \longrightarrow I$ is an isomorphism, and so $f = g$.

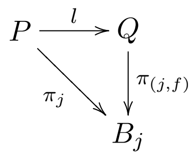
Now we want to construct a morphism $I \longrightarrow C$.
For $C \in \operatorname{ob}\mathcal{A}$, form the set $T = \{(j,f) |\ j \in J,\ f\colon C \longrightarrow B_j\}$, and the product $Q = \prod_{(j,f)} B_j$. We have a canonical morphism $h\colon C \longrightarrow Q$, defined by composition with the projections:



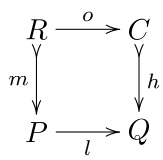
for all $(j,f) \in T$. This h is monic: for $D \overset{g_1}{\underset{g_2}{\rightrightarrows}} C \overset{h}{\longrightarrow} Q$ with $hg_1 = hg_2$, we have $fg_1 = fg_2$ for all $(j,f) \in T$.



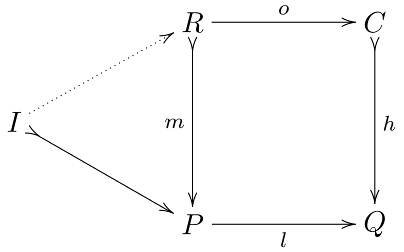
So as the B_j form a coseparating set, $g_1 = g_2$.
We also have a morphism $l\colon P \longrightarrow Q$ defined by



Form a pullback



Here m is also monic, as pullbacks preserve monos, so R is a subobject of P. But $I \rightarrowtail P$ is the smallest, so there is a morphism $I \rightarrowtail R$,



which gives a morphism $I \longrightarrow R \longrightarrow C$ as desired. □