Snake Lemma

Michaelmas 2011 Julia Goedecke

Theorem. *In an abelian category $\mathcal{A}$, a diagram*

$$\begin{array}{ccccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle a} & & \big\downarrow{\scriptstyle b} & & \big\downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & & \end{array}$$

with exact rows induces a six-term exact sequence

$$\operatorname{Ker} a \longrightarrow \operatorname{Ker} b \longrightarrow \operatorname{Ker} c \xrightarrow{\delta} \operatorname{Coker} a \longrightarrow \operatorname{Coker} b \longrightarrow \operatorname{Coker} c$$

between the kernels and cokernels.

Proof. Consider the kernels and cokernels with the induced maps between them. For shortness of notation we will write $\operatorname{Ker} a = K_1$, $\operatorname{Ker} b = K_2$ and $\operatorname{Ker} c = K_3$, similarly we will call the cokernels Q_i.

$$\begin{array}{ccccccccc} & & K_1 & \xrightarrow{\bar{f}} & K_2 & \xrightarrow{\bar{g}} & K_3 & & \\ & & \big\downarrow{\scriptstyle k_1} & & \big\downarrow{\scriptstyle k_2} & & \big\downarrow{\scriptstyle k_3} & & \\ & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle a} & & \big\downarrow{\scriptstyle b} & & \big\downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & & \\ & & \big\downarrow{\scriptstyle q_1} & & \big\downarrow{\scriptstyle q_2} & & \big\downarrow{\scriptstyle q_3} & & \\ & & Q_1 & \xrightarrow[\hat{f}]{} & Q_2 & \xrightarrow[\hat{g}]{} & Q_3 & & \end{array}$$

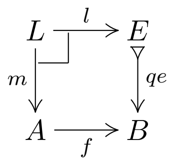
We give a proof which maximises the use of the Duality Principle (borrowed from Peter Johnstone).

1. Construction of δ Form the diagram

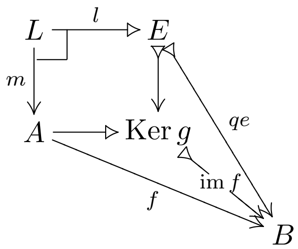
$$\begin{array}{ccccc} E & \overset{e}{\rightarrowtail} & P & \overset{p}{\twoheadrightarrow} & K_3 \\ & & {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle k_3} \\ & & B & \underset{g}{\twoheadrightarrow} & C \\ & & \big\downarrow{\scriptstyle b} & & \\ A' & \overset{f'}{\rightarrowtail} & B' & & \\ {\scriptstyle q_1}\big\downarrow & & \big\downarrow{\scriptstyle r} & & \\ Q_1 & \underset{t}{\rightarrowtail} & T & \underset{d}{\twoheadrightarrow} & D \end{array}$$

where the upper square is a pullback, the lower square is a pushout, $e = \ker p$ and $d = \operatorname{coker} t$. Remember that pullbacks and pushout preserve both monos and epis (as we are in an abelian category), so p and r are epis and q and t are monos. So as any epi is the cokernel of its kernel, we have $p = \operatorname{coker} e$

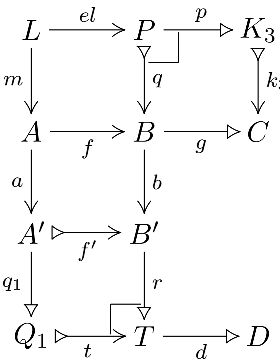
and dually $t = \ker d$. To construct $\delta\colon K_3 \longrightarrow C_1$, it is enough to factor the composite rbq through p and through t. For this we just have to show that $rbqe = 0$ and that $drbq = 0$, which are dual to each other, so showing the first is enough.
To prove the first, note that $gqe = k_3pe = 0$, so qe factors through $\ker g = \operatorname{im} f$. So if we form the pullback



then its top edge l is epic. This is because it is the same as the pullback:

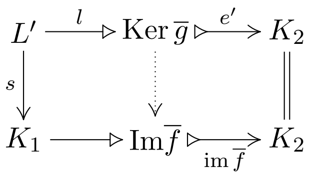


But $rbqel = rbfm = rf'am = tq_1am = 0$ (as q_1 is the cokernel of a),



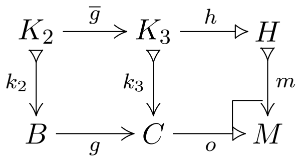
so we may deduce $rbqe = 0$ as required. So we get $\delta\colon K_3 \longrightarrow Q_1$ such that $t\delta p = rbq$.

Exactness at K_2 We have $k_3\overline{g}\overline{f} = gk_2\overline{f} = gfk_1 = 0$ and k_3 is monic, so $\overline{g}\overline{f} = 0$. Let $e'\colon E' \longrightarrow K_2$ be the kernel of $\overline{g}$; then the composite k_2e' factors through $\ker g = \operatorname{im} f$, so as before we get an epi $l'\colon L' \longrightarrow E'$ and a morphism $m'\colon L' \longrightarrow A$ such that $fm' = k_2e'l'$. Now $f'am' = bfm' = bk_2e'l' = 0$ and f' is monic, so $am' = 0$, i.e. m' factors through $\ker a = k_1$, say by $s\colon L' \longrightarrow K_1$. Now $k_2\overline{f}s = fk_1s = fm' = k_2e'l'$ and k_2 is monic, so $\overline{f}s = e'l'$, i.e. s is a morphism $e'l' \longrightarrow \overline{f}$ in $\mathcal{A}/K_2$. But this implies that $\operatorname{im}\overline{f} \geq \operatorname{im} e'l' = e' = \ker \overline{g}$ in $\operatorname{Sub}(K_2)$ (by naturality of image factorisation).

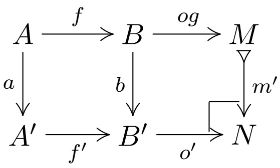


The reverse inequality follows from $\overline{g}\overline{f} = 0$, so we get exactness at K_2.

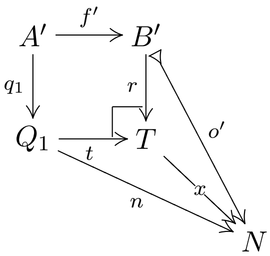
Exactness at K_3 The pair $(k_2, \overline{g})$ factors through the pullback P, say by $u\colon K_2 \longrightarrow P$. So to prove that $\delta\overline{g} = 0$, it suffices (since t is monic) to prove that $t\delta pu = 0$, i.e. that $rbqu = 0$ (since δ was induced by $t\delta p = rbq$). But this composite equals rbk_2, which is of course 0.
Now let $h\colon K_3 \longrightarrow H$ be the cokernel of $\overline{g}$, and form the pushout (the right-hand square)



where m is monic as k_3 is. Then $ogk_2 = ok_3\overline{g} = mh\overline{g} = 0$, so og factors through $\operatorname{coker} k_2 = \operatorname{coim} b$. So (as before with l) if we form another pushout (the right-hand square)



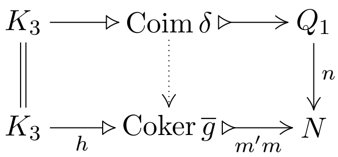
then m' is monic. Then $o'f'a = o'bf = m'ogf = 0$, so $o'f'$ factors through $\operatorname{coker} a = q_1$, say by $n\colon Q_1 \longrightarrow N$. Then the pair (o', n) factors through the pushout T, say by $x\colon T \longrightarrow N$.



Then

$$n\delta p = xt\delta p = xrbq = o'bq = m'ogq = m'ok_3p = m'mhp$$

and as p is epic, we have $n\delta = m'mh$, i.e. n is a morphism $\delta \longrightarrow m'mh$ in the coslice category $K_3\backslash\mathcal{A}$, so $\operatorname{coim}\delta \geq \operatorname{coim} m'mh = h = \operatorname{coker}\overline{g}$ in the preorder of quotients of K_3.



The reverse inequality follows from $\delta\overline{g} = 0$. So we have exactness at K_3.

Exactness at Q_1 and Q_2 These proofs are dual to those at K_3 and K_2 respectively. □