# HOMOLOGY IN RELATIVE SEMI-ABELIAN CATEGORIES

JULIA GOEDECKE

Abstract. We use recent results on simplicial objects in relative Mal'tsev categories and a classical comparison theorem to obtain homology with coefficients in a relative semi-abelian category as defined by T. Janelidze.

Contents



## 1. Introduction

In a recent paper [6] with T. Everaert and T. Van der Linden, we define relative Mal'tsev categories and show amongst other things that in such a category every simplicial object satisfies a relative version of the Kan property. Here we use this result to define homology with coefficients in a relative semi-abelian category, as defined by T. Janelidze [14].

**Resolutions in terms of exactness.** We first study simplicial resolutions in the context of relative homological categories. Section 2 gives the relevant definitions, in particular defining an $\mathcal{E}$-simplicial object to be one where all face operators are in the class $\mathcal{E}$ which makes the theory relative. In Section 3 we compare conditions for an augmented simplicial object $\mathbb{A}$ to be a resolution to exactness conditions on the Moore complex of $\mathbb{A}$. We define a chain complex to be *$\mathcal{E}$-exact* when the factorisations to the kernels are morphisms in $\mathcal{E}$, in order to prove Theorem 3.9:

> *an augmented $\mathcal{E}$-simplicial object is an $\mathcal{E}$-resolution if and only if its Moore normalisation is $\mathcal{E}$-exact.*

This is often used in the absolute case of homological categories and appears in some form in [18], but as far as we know it has not been stated in exactly this form before. When the pair $(\mathcal{A}, \mathcal{E})$ is relatively semi-abelian, we can rephrase the above result to say that

> *an augmented $\mathcal{E}$-simplicial object $\mathbb{A}$ is an $\mathcal{E}$-resolution if and only if* $\mathrm{H}_n\mathbb{A} = 0$ *for* $n \geq 1$ *and* $\mathrm{H}_0\mathbb{A} = A_{-1}$

(Theorem 4.11). Furthermore, any short $\mathcal{E}$-exact sequence of $\mathcal{E}$-simplicial objects induces a long $\mathcal{E}$-exact homology sequence (Theorem 4.12). Analogously to the absolute case, this translation into homology crucially relies on the fact that the Moore complex of an $\mathcal{E}$-simplicial object in a relative

---

*Date*: 4 July 2011.

1991 *Mathematics Subject Classification.* 18A20, 18E10, 18G25, 18G50, 20J.

*Key words and phrases.* homology; simplicial resolution; Mal'tsev condition; relative homological algebra; homological, semi-abelian category.

Supported by the FNRS grant *Crédit aux chercheurs* 1.5.016.10F and Queens' College, Cambridge.

semi-abelian category is $\mathcal{E}$-proper (Lemma 4.9), meaning that every morphism in it factors as an extension followed by an $\mathcal{E}$-normal monomorphism, a kernel of an extension.

**Homology functors.** In Section 5 we prove that in a category $(\mathcal{A}, \mathcal{E})$ satisfying (E1)–(E3) (see Definition 2.1), two projective $\mathcal{E}$-resolutions of an object $A$ give rise to the same homology with coefficients in a functor $I\colon \mathcal{A} \to \mathcal{B}$ to a relative semi-abelian category $(\mathcal{B}, \mathcal{F})$. This makes it possible to define the homology of an object, as in Definition 5.12, and thus obtain homology functors

$$\mathrm{H}_{n+1}(-, I)\colon \mathcal{A} \to \mathcal{B}.$$

As in the classical situation we go via the fact that homotopic morphisms between simplicial objects give rise to the same homology (Corollary 5.6). To prove this result, we show that the kernel of the projection of a cocylinder object $\epsilon_0\colon \mathbb{A}^I \to \mathbb{A}$ is an $\mathcal{F}$-exact $\mathcal{F}$-simplicial object (Proposition 5.5), making use of the fact that every $\mathcal{F}$-simplicial object in the relative semi-abelian category $(\mathcal{B}, \mathcal{F})$ is $\mathcal{F}$-Kan.

## 2. Resolutions in relative homological categories

We first give some necessary definitions and recall useful results.

**Relative homological categories.** The first part of this paper is set in the context of *relative homological categories* as defined by T. Janelidze [12]. We make only a small change: we do not need the existence of all cokernels.

**Definition 2.1.** Let $\mathcal{A}$ be a pointed finitely complete category and let $\mathcal{E}$ be a class of normal epimorphisms in $\mathcal{A}$. The pair $(\mathcal{A}, \mathcal{E})$ is called a **relative homological category** if it satisfies the following axioms:

(E1) $\mathcal{E}$ contains all isomorphisms;
(E2) $\mathcal{E}$ is pullback-stable;
(E3) $\mathcal{E}$ is closed under composition;
(E4) if $f \in \mathcal{E}$ and $g \circ f \in \mathcal{E}$ then $g \in \mathcal{E}$;
(E5) given a diagram in $\mathcal{A}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & 0 \\ & & {\scriptstyle k}\downarrow & & {\scriptstyle a}\downarrow & & \| & & \\ 0 & \longrightarrow & K' & \longrightarrow & A' & \xrightarrow[f']{} & B & \longrightarrow & 0 \end{array}$$

with short exact rows and $f$ and $f'$ in $\mathcal{E}$, if $k \in \mathcal{E}$ then also $a \in \mathcal{E}$.

(F) if a morphism $f$ in $\mathcal{A}$ factors as $f = e \circ m$ with $m$ a monomorphism and $e \in \mathcal{E}$, then it also factors (essentially uniquely) as $f = m' \circ e'$ with $m'$ a monomorphism and $e' \in \mathcal{E}$.

**Example 2.2.** When $\mathcal{E}$ is the class of all regular epimorphisms in a regular category $\mathcal{A}$, then $(\mathcal{A}, \mathcal{E})$ is a relative homological category if and only if $\mathcal{A}$ is homological.

Any relative semi-abelian category (Definition 4.6) is a relative homological category. For example, if $\mathcal{A}$ is semi-abelian and $\mathcal{E}$ is the class of central extensions in the sense of Huq, closed under composition, then $(\mathcal{A}, \mathcal{E})$ is relatively semi-abelian [13, Proposition 5.3.2]. That is, any morphism in $\mathcal{E}$ is a composite of regular epimorphisms $f\colon A \to B$ with $[\mathrm{K}[f], A] = 0$, where $[\mathrm{K}[f], A]$ is the commutator of $\mathrm{K}[f]$ and $A$ in the sense of Huq [9]. Another relative homological category $(\mathcal{A}, \mathcal{E})$ is given by taking $\mathcal{E}$ to be the *trivial extensions* as defined by categorical Galois theory [10] induced by a Birkhoff subcategory $\mathcal{B}$ of a semi-abelian category $\mathcal{A}$. This will be explained by T. Janelidze in her forthcoming paper *Central extensions generate a relative semi-abelian category structure.*

Morphisms in the class $\mathcal{E}$ will be called **extensions**. We write $\mathsf{Ext}\mathcal{A}$ for the full subcategory of the category $\mathsf{Arr}\mathcal{A}$ of arrows in $\mathcal{A}$ determined by the elements of $\mathcal{E}$. Any such class of extensions $\mathcal{E}$ gives rise to *double extensions*, which were first defined in the context of groups in [11, 4] and appeared in a categorical context in [2, 7, 5].

**Definition 2.3.** Let $\mathcal{E}$ be a class of extensions satisfying axioms (E1)–(E3). A **double extension** is a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow[f']{} & B' \end{array}$$

in $\mathcal{A}$ where $a$, $b$, $f$, $f'$ and the induced morphism $\langle a, f\rangle\colon A \to A' \times_{B'} B$ to the pullback of $b$ and $f'$ are extensions. We denote the class of double extensions thus obtained by $\mathcal{E}^1$.

Any extension, being a normal epimorphism, is the cokernel of its kernel. Hence the following relative concept of short exact sequence makes sense [12].

**Definition 2.4.** A morphism in a relative homological category $(\mathcal{A}, \mathcal{E})$ is an **$\mathcal{E}$-normal monomorphism** when it is the kernel of an extension. A **short $\mathcal{E}$-exact sequence** consists of an $\mathcal{E}$-normal monomorphism followed by its cokernel (an element of $\mathcal{E}$), and is usually denoted

$$0 \longrightarrow K \longrightarrow A \xrightarrow{f} B \longrightarrow 0. \qquad (\mathbf{A})$$

For any type of graded object we use this notion in the degreewise sense.

$\mathcal{E}$-normal monomorphisms share various stability properties with normal monomorphisms. For instance:

**Lemma 2.5.** *In a relative homological category, if an $\mathcal{E}$-normal monomorphism $k$ factors as a morphism $l$ followed by a monomorphism $m$, then $l$ is an $\mathcal{E}$-normal monomorphism.*

*Proof.* If $k$ is a kernel of $e \in \mathcal{E}$ then $l$ is a kernel of $e' \in \mathcal{E}$, where $m' \circ e' = e \circ m$ is the factorisation which exists by (F). □

We also obtain the following characterisation of double extensions.

**Proposition 2.6** ([5, Lemma 1.7])**.** *Let $(\mathcal{A}, \mathcal{E})$ satisfy (E1)–(E3) and (E5). Given a diagram in $\mathcal{A}$ with short $\mathcal{E}$-exact rows and $a$ and $b$ in $\mathcal{E}$*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & 0 \\ & & {\scriptstyle k}\downarrow & & {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \\ 0 & \longrightarrow & K' & \longrightarrow & A' & \xrightarrow[f']{} & B' & \longrightarrow & 0, \end{array} \qquad (\mathbf{B})$$

*then $k$ is in $\mathcal{E}$ if and only if the right hand square is a double extension.*

*Proof.* The right-to-left implication follows from (E2), and the other direction uses (E5). □

We also need the following basic result which is well known to hold in the absolute case [1, Proposition 7]. Its non-trivial implication is an immediate consequence of the $\mathcal{E}$-Short Five Lemma [12], which itself follows from (E5) and (E1).

**Proposition 2.7.** *Let $(\mathcal{A}, \mathcal{E})$ satisfy (E1)–(E3) and (E5). In a diagram such as* (**B**) *with short $\mathcal{E}$-exact rows, $k$ is an isomorphism if and only if the right hand square is a pullback.* □

Note that for this proposition, we do not require the morphisms $a$ and $b$ in the diagram (**B**) to be extensions.

**Resolutions.** To define resolutions, which will play a crucial role throughout the paper, we first have to define simplicial objects and the weaker semi- and quasi-simplicial objects.

**Definition 2.8** (Augmented semi-simplicial objects)**.** Let $\Delta_s$ be the category which has as objects the finite ordinals $n \geq 0$ and as morphisms injective order-preserving maps. We may call it the *augmented semi-simplicial category.* Given a category $\mathcal{A}$, an **(augmented) semi-simplicial object** in $\mathcal{A}$ is a functor

$$\mathbb{A}\colon (\Delta_s)^{\mathrm{op}} \to \mathcal{A}.$$

We denote the objects $\mathbb{A}(n)$ by $A_{n-1}$, and the image of the inclusion $n \to n+1$ which leaves out $i$ by $\partial_i$, so that an augmented semi-simplicial object $\mathbb{A}$ corresponds to the following data: a sequence of objects $(A_n)_{n\geq -1}$ with face operators (or faces) $(\partial_i\colon A_n \to A_{n-1})_{0\leq i\leq n}$ for $0 \leq n$,

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; A_2 \;\underset{\partial_2}{\overset{\partial_0}{\substack{\longrightarrow\\\xrightarrow{\partial_1}\\\longrightarrow}}}\; A_1 \;\underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}}\; A_0 \xrightarrow{\partial_0} A_{-1}$$

subject to the identity

$$\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$$

for $i < j$. The morphism $\partial_0\colon A_0 \to A_{-1}$ is called the **augmentation** of $\mathbb{A}$.

We write $\mathsf{S}_s\mathcal{A}$ for the category of augmented semi-simplicial objects in $\mathcal{A}$, which is of course the functor category $\mathsf{Fun}((\Delta_s)^{\mathrm{op}}, \mathcal{A})$.

**Definition 2.9** (Augmented quasi-simplicial objects)**.** We can add degeneracy operators (or degeneracies) $(\sigma_i\colon A_n \to A_{n+1})_{0\leq i\leq n}$ for $0 \leq n$ to an (augmented) semi-simplicial object to obtain an **(augmented) quasi-simplicial object** $\mathbb{A}$ in $\mathcal{A}$, satisfying the identities

$$\partial_i \circ \sigma_j = \begin{cases} \sigma_{j-1} \circ \partial_i & \text{if } i < j \\ 1 & \text{if } i = j \text{ or } i = j+1 \\ \sigma_j \circ \partial_{i-1} & \text{if } i > j+1. \end{cases}$$

The augmented quasi-simplicial objects in $\mathcal{A}$ with the natural morphisms between them form a category $\mathsf{S}_q\mathcal{A}$ which may be seen as a functor category $\mathsf{Fun}((\Delta_q)^{\mathrm{op}}, \mathcal{A})$.

**Definition 2.10.** If the degeneracies in an augmented quasi-simplicial object $\mathbb{A}$ also satisfies

$$\sigma_i \circ \sigma_j = \sigma_{j+1} \circ \sigma_i$$

for all $i \leq j$, then $\mathbb{A}$ is an **(augmented) simplicial object**. Of course this is just a functor from $\Delta^{\mathrm{op}}$ to $\mathcal{A}$ where $\Delta$ is the category of finite ordinals with all order-preserving maps.

Note that this notation of including 0 in $\Delta$ agrees with MacLane [15], though algebraic topologists usually don't include 0 because they talk about non-augmented simplicial objects. This is the reason for the numbering shift $\mathbb{A}(n) = A_{n-1}$.

In the course of the paper we will need the following induced semi-simplicial object.

**Notation 2.11.** Let $\mathbb{A}$ be an augmented semi-simplicial object in $\mathcal{A}$. It induces another augmented semi-simplicial object $\mathbb{A}^-$ with

$$A^-_{n-1} = A_n \qquad \text{and} \qquad \partial^-_i = \partial_{i+1}\colon A_{n+1} \to A_n,$$

for $n \geq 0$ and $0 \leq i \leq n$. This is the augmented semi-simplicial object obtained from $\mathbb{A}$ by leaving out $A_{-1}$ and all $\partial_0\colon A_n \to A_{n-1}$. The left out $\partial_0$ combine to give a morphism $\partial = (\partial_0)_n$ from $\mathbb{A}^-$ to $\mathbb{A}$.

When $\mathbb{A}$ is a (quasi)-simplicial object, the degeneracy operators can be shifted in the same way to give a (quasi)-simplicial object $\mathbb{A}^-$ and a morphism $\partial\colon \mathbb{A}^- \to \mathbb{A}$ of (quasi)-simplicial objects.

As we are dealing with a class of extensions $\mathcal{E}$, we are mainly interested in the following special (semi)-simplicial objects.

**Definition 2.12.** An **(augmented) $\mathcal{E}$-semi-simplicial object** $\mathbb{A}$ is an (augmented) semi-simplicial object with all faces $\partial_i$ in $\mathcal{E}$. We write $\mathsf{S}_s(\mathcal{A},\mathcal{E})$ for the induced category and $\mathsf{S}_q(\mathcal{A},\mathcal{E})$ for the category of augmented $\mathcal{E}$-quasi-simplicial objects.

In this context, Proposition 2.6 implies

**Proposition 2.13.** *Let $(\mathcal{A},\mathcal{E})$ satisfy (E1)–(E3) and (E5). Given a morphism of (augmented) $\mathcal{E}$-semi-simplicial objects $f$ which is degreewise in $\mathcal{E}$, consider the induced short $\mathcal{E}$-exact sequence of (augmented) semi-simplicial objects*

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathbb{A} \xrightarrow{\ f\ } \mathbb{B} \longrightarrow 0. \tag{\textbf{C}}$$

*The kernel $\mathbb{K}$ is an (augmented) $\mathcal{E}$-semi-simplicial object if and only if $f$ is an (augmented) $\mathcal{E}^1$-semi-simplicial object in $\mathsf{Ext}\mathcal{A}$.* □

To define resolutions, we will use the notion of a simplicial kernel.

**Definition 2.14** (Simplicial kernels)**.** Let

$$(f_i\colon X \to Y)_{0\leq i\leq n}$$

be a sequence of $n+1$ morphisms in the category $\mathcal{A}$. A **simplicial kernel** of $(f_0,\ldots,f_n)$ is a sequence

$$(k_i\colon K \to X)_{0\leq i\leq n+1}$$

of $n+2$ morphisms in $\mathcal{A}$ satisfying $f_i k_j = f_{j-1} k_i$ for $0 \leq i < j \leq n+1$, which is universal with respect to this property. In other words, it is the limit for a certain diagram in $\mathcal{A}$, giving a universal object satisfying the (semi)-simplicial identities.

For example, the simplicial kernel of one morphism is just its kernel pair. If $\mathcal{A}$ has all pullbacks, then simplicial kernels exist, as they can be formed by successive pullbacks (see, for instance, [16]).

**Definition 2.15.** An (augmented) semi-simplicial object $\mathbb{A}$ is said to be **$\mathcal{E}$-exact at** $A_{n-1}$ when the simplicial kernel $K_n\mathbb{A}$ of $(\partial_i\colon A_{n-1} \to A_{n-2})_{0\leq i\leq n-1}$ exists and the factorisation $A_n \to K_n\mathbb{A}$ is in $\mathcal{E}$. (Here we also write $K_0\mathbb{A} = A_{-1}$, i.e. $\mathbb{A}$ is $\mathcal{E}$-exact at $A_{-1}$ if $\partial_0\colon A_0 \to A_{-1}$ is in $\mathcal{E}$.)

An augmented semi-simplicial object $\mathbb{A}$ is called an **$\mathcal{E}$-resolution (of $A_{-1}$)** when $\mathbb{A}$ is $\mathcal{E}$-exact at $A_n$ for all $n \geq -1$.

Notice that an $\mathcal{E}$-resolution is always an $\mathcal{E}$-semi-simplicial object. There is a connection between $\mathcal{E}$-resolutions and $\mathcal{E}^1$-resolutions:

**Lemma 2.16** ([6, Corollary 2.19])**.** *An augmented semi-simplicial object $\mathbb{A}$ is an $\mathcal{E}$-resolution if and only if the augmented semi-simplicial object of arrows $\partial\colon \mathbb{A}^- \to \mathbb{A}$ is an $\mathcal{E}^1$-resolution.* □

$\mathcal{E}$-resolutions are studied in more depth in the paper [6].

## 3. Resolutions via the Moore complex

The aim of this section is to analyse simplicial resolutions in terms of an associated chain complex. For the whole section, $(\mathcal{A},\mathcal{E})$ will be a relative homological category, if not explicitly stated otherwise.

In semi-abelian categories, the *Moore complex* or *normalised chain complex* of a simplicial object is commonly used to detect exactness of the given simplicial object (see, for instance, [8]). In our setting we use the notion of $\mathcal{E}$-exactness for a chain complex and are thus able to link this to $\mathcal{E}$-exact $\mathcal{E}$-simplicial objects.

**Definition 3.1.** A **chain complex** $C$ in $\mathcal{A}$ is a collection of morphisms

$$(d_n\colon C_n \to C_{n-1})_{n\geq 0}$$

such that $d_n \circ d_{n+1} = 0$, for all $n \geq 0$. The category of chain complexes in $\mathcal{A}$ (with, as morphisms, the obvious commutative diagrams) is denoted by $\mathsf{Ch}\mathcal{A}$.

Let $C$ be a chain complex and $n \geq 0$. $C$ is said to be **$\mathcal{E}$-exact at** $C_n$ when the factorisation of $d_{n+1}$ over the kernel $\mathrm{K}[d_n]$ of $d_n$ is in $\mathcal{E}$.

$$\begin{array}{ccccccc} \cdots \to C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} \to \cdots \\ & {\scriptstyle d'_{n+1}\in\mathcal{E}} \searrow & \nearrow {\scriptstyle \operatorname{Ker} d_n} & & \\ & & \mathrm{K}[d_n] & & \end{array}$$

We also say that $C$ is **$\mathcal{E}$-exact at** $C_{-1}$ when $d_0$ is in $\mathcal{E}$. The chain complex $C$ is **$\mathcal{E}$-exact** when it is exact at $C_n$ for all $n \geq -1$.

**Remark 3.2.** Our definition of exactness at $C_{-1}$ is equivalent to $C_0 \to C_{-1} \to 0$ being $\mathcal{E}$-exact at $C_{-1}$.

For instance, a sequence of morphisms such as (**A**) above where the middle pair composes to zero is a short $\mathcal{E}$-exact sequence if and only if it is an $\mathcal{E}$-exact chain complex, which justifies the notation for short $\mathcal{E}$-exact sequences. (More precisely, its exactness at $B$ means that $f \in \mathcal{E}$, exactness at $A$ says that $K \to A$ factors as an extension followed by a kernel of $f$, and exactness at $K$ ensures that this extension is an isomorphism.)

**Definition 3.3** (Moore normalisation)**.** Let $(\mathcal{A}, \mathcal{E})$ be a relative homological category. The **normalisation functor**

$$\mathrm{N}\colon \mathsf{S}_s(\mathcal{A}, \mathcal{E}) \to \mathsf{Ch}\mathcal{A}$$

turns an augmented $\mathcal{E}$-semi-simplicial object $\mathbb{A}$ into the **Moore complex** $\mathrm{N}\mathbb{A}$ of $\mathbb{A}$, the chain complex with $\mathrm{N}_{-1}\mathbb{A} = A_{-1}$, $\mathrm{N}_0\mathbb{A} = A_0$,

$$\mathrm{N}_n\mathbb{A} = \bigcap_{i=0}^{n-1} \mathrm{K}[\partial_i \colon A_n \to A_{n-1}]$$

and differentials

$$d_n = \partial_n \circ \bigcap_i \operatorname{Ker} \partial_i \colon \mathrm{N}_n\mathbb{A} \to \mathrm{N}_{n-1}\mathbb{A}$$

for $n \geq 1$, and $d_0 = \partial_0 \colon A_0 \to A_{-1}$. That is, $d_n$ is the morphism induced by $\partial_n$ via

$$\begin{array}{ccc} N_n\mathbb{A} & \xrightarrow{d_n} & N_{n-1}\mathbb{A} \\ \downarrow & & \downarrow \\ A_n & \xrightarrow[\partial_n]{} & A_{n-1}. \end{array}$$

Notice that the Moore complex defined here differs very slightly from its usual form: we have added an object $N_{-1}\mathbb{A}$, such that the augmentation $A_0 \to A_{-1}$ of an augmented $\mathcal{E}$-semi-simplicial object appears in the complex. This makes it easier to compare $\mathcal{E}$-resolutions and their Moore complex in a context not using homology. However, when we can use homology, as in Definition 4.10, this addition to the Moore complex is not needed.

We now recall a very useful tool for working with the Moore complex of a simplicial object.

**Notation 3.4.** Let $\mathbb{A}$ be an augmented $\mathcal{E}$-semi-simplicial object and recall the definition of $\mathbb{A}^-$ from Notation 2.11. The kernel of $\partial\colon \mathbb{A}^- \to \mathbb{A}$ is denoted by $\Lambda\mathbb{A}$. Thus we obtain a short $\mathcal{E}$-exact sequence of augmented semi-simplicial objects:

$$0 \longrightarrow \Lambda\mathbb{A} \longrightarrow \mathbb{A}^- \xrightarrow{\partial} \mathbb{A} \longrightarrow 0$$

**Remark 3.5.** By Proposition 2.13, the augmented semi-simplicial object $\Lambda\mathbb{A}$ is an augmented $\mathcal{E}$-semi-simplicial object if and only if $\partial\colon \mathbb{A}^- \to \mathbb{A}$ is an augmented $\mathcal{E}^1$-semi-simplicial object in $\mathsf{Ext}\mathcal{A}$. Similarly, since taking kernels commutes with taking simplicial kernels, Proposition 2.6 tells us that $\Lambda\mathbb{A}$ is an $\mathcal{E}$-resolution if and only if $\partial\colon \mathbb{A}^- \to \mathbb{A}$ is an $\mathcal{E}^1$-resolution, which by Lemma 2.16 is equivalent to $\mathbb{A}$ being an $\mathcal{E}$-resolution.

The following lemma is classical in the absolute setting and follows easily from the definition.

**Lemma 3.6.** *Let $(\mathcal{A}, \mathcal{E})$ be a relative homological category. If $\partial\colon \mathbb{A}^- \to \mathbb{A}$ is an $\mathcal{E}^1$-semi-simplicial object then*

$$\mathrm{N}_{k-1}\Lambda\mathbb{A} = \mathrm{N}_k\mathbb{A} \quad \textit{and} \quad d_k^{\Lambda\mathbb{A}} = d_{k+1}^{\mathbb{A}}$$

*for all $k \geq 1$.* □

**Remark 3.7.** If $\mathbb{A}$ is an $\mathcal{E}$-quasi-simplicial object, then $\partial\colon \mathbb{A}^- \to \mathbb{A}$ is an $\mathcal{E}^1$-semi-simplicial object, which follows from the *relative Mal'tsev axiom* given in [6], and the lemma above applies. (The numbering of axioms in [6] differs slightly from the one here, using (E5) for the relative Mal'tsev axiom and (E5$^+$) for our (E5).) Notice that we do not need the augmentation of an augmented quasi-simplicial object to be in $\mathcal{E}$ resp. $\mathcal{E}^1$ for this lemma.

When restricted to $\mathcal{E}$-quasi-simplicial objects, the Moore normalisation functor is exact. This is a crucial result for the long exact homology sequence in the following section.

**Proposition 3.8.** *The normalisation functor*

$$\mathrm{N}\colon \mathsf{S}_q(\mathcal{A}, \mathcal{E}) \to \mathsf{Ch}\mathcal{A}$$

*is exact: it sends short $\mathcal{E}$-exact sequences of augmented $\mathcal{E}$-quasi-simplicial objects to short $\mathcal{E}$-exact sequences of chain complexes in $\mathcal{A}$.*

*Proof.* Let $(\mathbf{C})$ be a short $\mathcal{E}$-exact sequence of augmented $\mathcal{E}$-quasi-simplicial objects. Then for $n \in \{-1, 0\}$, the sequence

$$0 \longrightarrow \mathrm{N}_n\mathbb{K} \longrightarrow \mathrm{N}_n\mathbb{A} \xrightarrow{\mathrm{N}_n f} \mathrm{N}_n\mathbb{B} \longrightarrow 0$$

is already short $\mathcal{E}$-exact in $\mathcal{A}$. Using the relative 3×3-Lemma [12, Lemma 4.3] degreewise on the diagram of $\mathcal{E}$-quasi-simplicial objects

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda\mathbb{K} & \longrightarrow & \Lambda\mathbb{A} & \xrightarrow{\Lambda f} & \Lambda\mathbb{B} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{K}^- & \longrightarrow & \mathbb{A}^- & \xrightarrow{f^-} & \mathbb{B}^- & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
0 & \longrightarrow & \mathbb{K} & \longrightarrow & \mathbb{A} & \xrightarrow[f]{} & \mathbb{B} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

we see that the top row is also a short $\mathcal{E}$-exact sequence. (Notice that $\Lambda\mathbb{K}$, $\Lambda\mathbb{A}$ and $\Lambda\mathbb{B}$ are indeed $\mathcal{E}$-quasi-simplicial objects by Proposition 2.13, as $\mathbb{K}$, $\mathbb{A}$ and $\mathbb{B}$ are $\mathcal{E}$-quasi-simplicial objects and so each $\partial$ is an $\mathcal{E}^1$-semi-simplicial object.) The result now follows by induction via Lemma 3.6. □

Lemma 3.6 also allows us to use induction in the next theorem, our main result of this section.

**Theorem 3.9.** *Let $(\mathcal{A}, \mathcal{E})$ be a relative homological category. An augmented $\mathcal{E}$-(quasi)-simplicial object $\mathbb{A}$ in $\mathcal{A}$ is an $\mathcal{E}$-resolution if and only if its normalisation* $\mathrm{N}\mathbb{A}$ *is $\mathcal{E}$-exact.*

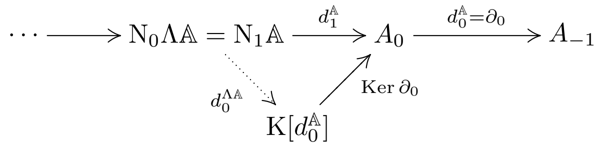
*Proof.* ⇒ Given any $\mathcal{E}$-resolution $\mathbb{A}$, the morphism

$$d_0 = \partial_0\colon \mathrm{N}_0\mathbb{A} = A_0 \longrightarrow A_{-1} = \mathrm{N}_{-1}\mathbb{A}$$

is always an extension, which gives $\mathcal{E}$-exactness at $N_{-1}\mathbb{A}$. Since $\Lambda\mathbb{A}$ is also an $\mathcal{E}$-resolution,

$$d_0^{\Lambda\mathbb{A}}\colon \mathrm{N}_0\Lambda\mathbb{A} = \mathrm{N}_1\mathbb{A} \longrightarrow \mathrm{K}[d_0^{\mathbb{A}}] = \mathrm{N}_{-1}\Lambda\mathbb{A}$$

is in $\mathcal{E}$, which means that $\mathrm{N}\mathbb{A}$ is $\mathcal{E}$-exact at $\mathrm{N}_0\mathbb{A}$.




Now suppose for some $n \geq 1$ that, given any $\mathcal{E}$-resolution $\mathbb{A}$, the chain complex $\mathrm{N}\mathbb{A}$ is $\mathcal{E}$-exact at $\mathrm{N}_{n-1}\mathbb{A}$, i.e., the factorisation of $d_n \colon \mathrm{N}_n\mathbb{A} \to \mathrm{N}_{n-1}\mathbb{A}$ over $\mathrm{K}[d_{n-1}]$ is in $\mathcal{E}$. In particular, this is true for $\Lambda\mathbb{A}$, so using Lemma 3.6 the factorisation of

$$d_n^{\Lambda\mathbb{A}} = d_{n+1}^{\mathbb{A}} \colon \mathrm{N}_n\Lambda\mathbb{A} = \mathrm{N}_{n+1}\mathbb{A} \longrightarrow \mathrm{N}_n\mathbb{A} = \mathrm{N}_{n-1}\Lambda\mathbb{A}$$

over $\mathrm{K}[d_{n-1}^{\Lambda\mathbb{A}}] = \mathrm{K}[d_n^{\mathbb{A}}]$ is in $\mathcal{E}$. Hence the result holds by induction.

$\Leftarrow$ If $\mathrm{N}\mathbb{A}$ is $\mathcal{E}$-exact then

$$\partial_0 = d_0 \colon \mathrm{N}_0\mathbb{A} = A_0 \longrightarrow A_{-1} = \mathrm{N}_{-1}\mathbb{A}$$

is in $\mathcal{E}$ by exactness at $N_{-1}\mathbb{A}$. Now assume for $n \geq 0$ that if $\mathrm{N}\mathbb{A}$ is $\mathcal{E}$-exact then $\mathbb{A}$ is an $\mathcal{E}$-resolution up to level $n$. In particular, $\Lambda\mathbb{A}$ is an $\mathcal{E}$-resolution up to level $n$, since $\mathrm{N}\Lambda\mathbb{A}$ is also $\mathcal{E}$-exact by Lemma 3.6 (noting also the relationship between $d_1^{\mathbb{A}}$ and $d_0^{\Lambda\mathbb{A}}$ shown above). Using Remark 3.5 and Lemma 2.16, this implies that $\mathbb{A}$ is an $\mathcal{E}$-resolution up to level $n+1$, so the result holds by induction. □

## 4. Homology of simplicial objects

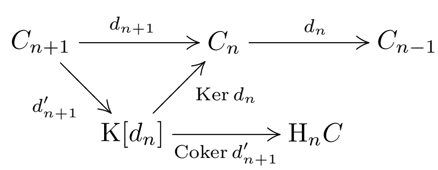
As in the absolute case of semi-abelian categories, we will define homology of simplicial objects by going via the Moore complex. We now go through the separate steps needed for this definition.

**Homology of $\mathcal{E}$-proper chain complexes.** Analogously to homology of proper chain complexes in homological categories, we can define homology for $\mathcal{E}$-proper chain complexes.

**Definition 4.1.** A morphism in a relative homological category $(\mathcal{A}, \mathcal{E})$ is **$\mathcal{E}$-proper** when it factors as an extension followed by an $\mathcal{E}$-normal monomorphism. A chain complex $C$ in $(\mathcal{A}, \mathcal{E})$ is **$\mathcal{E}$-proper** when the morphism $d_n \colon C_n \to C_{n-1}$ is $\mathcal{E}$-proper for every $n \geq 0$.

For instance, an $\mathcal{E}$-exact chain complex is always $\mathcal{E}$-proper.

**Definition 4.2** (Homology of $\mathcal{E}$-proper chain complexes)**.** Given an $\mathcal{E}$-proper chain complex $C$ in $(\mathcal{A}, \mathcal{E})$ and $n \geq 0$, we define the **$n$th homology object** $\mathrm{H}_nC$ to be the cokernel of the factorisation $d'_{n+1}$ of $d_{n+1}$ over $\mathrm{K}[d_n]$.

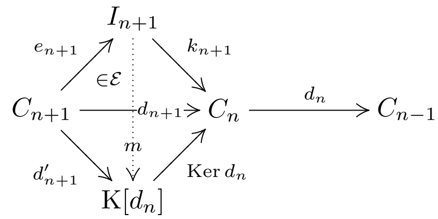



Note that the cokernel of $d'_{n+1}$ indeed exists and is in fact an extension — to see this, one uses Lemma 2.5 as in the following proof. Of course, this definition, which is exactly the same as in the absolute case, only makes sense here if it detects $\mathcal{E}$-exactness of the complex, which we now prove:

**Proposition 4.3.** *An $\mathcal{E}$-proper chain complex $C$ is $\mathcal{E}$-exact at $C_n$ if and only if $\mathrm{H}_nC$ is zero.*

*Proof.* As $C$ is $\mathcal{E}$-proper, we can factor $d_{n+1}$ as an extension $e_{n+1}$ followed by an $\mathcal{E}$-normal monomorphism $k_{n+1}$. As any extension is a (normal) epimorphism, this $\mathcal{E}$-normal monomorphism

$k_{n+1}$ factors over $\mathrm{K}[d_n]$ by a monomorphism $m$.

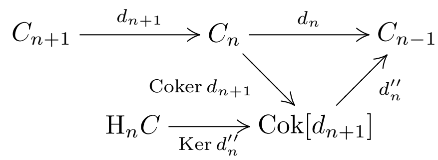



By Lemma 2.5, $m$ is an $\mathcal{E}$-normal monomorphism. Note that we also have $m \circ e_{n+1} = d'_{n+1}$ as $\mathrm{Ker}\, d_n$ is a monomorphism. As $e_{n+1}$ is a (normal) epimorphism, the cokernel of $m$ (an extension) is the same as the cokernel of $d'_{n+1}$, and furthermore $m$ is the kernel of its cokernel. Hence $\mathrm{H}_n C = 0$ if and only if $m$ is an isomorphism, and thus if and only if $C$ is $\mathcal{E}$-exact at $C_n$. □

Note that this proposition does not quite follow from the absolute case: as the definition of homology is the same and any $\mathcal{E}$-proper chain complex $C$ is certainly proper, the result in the absolute case implies that an $\mathcal{E}$-proper chain complex is exact at $C_n$ if and only if $\mathrm{H}_n C$ is zero. However, in this result we show that it is in fact $\mathcal{E}$-exact, which is more restrictive than just exact.

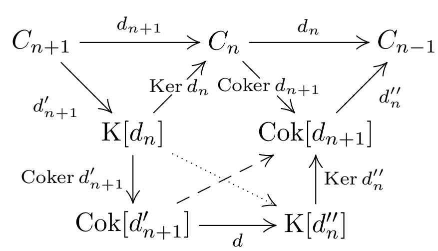
As in the absolute case [8, Proposition 2.4], any short exact sequence of proper chain complexes induces a long exact homology sequence. For this we first introduce the dual definition of homology of a chain complex.

**Lemma 4.4.** *Given an $\mathcal{E}$-proper chain complex $C$ in $(\mathcal{A}, \mathcal{E})$,*




*the nth homology object $\mathrm{H}_n C$ of $C$ may also be obtained as the kernel of the factorisation $d''_n$ of $d_n$ over $\mathrm{Cok}[d_{n+1}]$.*

*Proof.* In the diagram




which displays the two constructions, the induced comparison morphism $d$ between them is an isomorphism. Indeed, the dotted arrow induced by taking kernels is an extension, as it is a pullback of the extension $\mathrm{Coker}\, d_{n+1}$, so $d$ is an extension by (E4). On the other hand, taking cokernels induces the broken arrow in the diagram, and the thus obtained commutative square is a pullback by Proposition 2.7 (recall from the proof of Proposition 4.3 that $\mathrm{K}[\mathrm{Coker}\, d'_{n+1}] = \mathrm{K}[\mathrm{Coker}\, d_{n+1}]$). As a consequence, the kernel of the extension $d$ is zero, being isomorphic to the kernel of $\mathrm{Ker}\, d_n$. This again uses Proposition 2.7, this time in the other direction. □

**Proposition 4.5** (Long exact homology sequence)**.** *In a relative homological category $(\mathcal{A}, \mathcal{E})$, any short $\mathcal{E}$-exact sequence of $\mathcal{E}$-proper chain complexes*

$$0 \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow 0$$

*gives rise to a long $\mathcal{E}$-exact homology sequence*

$$\cdots \longrightarrow \mathrm{H}_{n+1}C \longrightarrow \mathrm{H}_{n+1}D \longrightarrow \mathrm{H}_{n+1}E \longrightarrow \mathrm{H}_n C \longrightarrow \cdots \longrightarrow \mathrm{H}_0 E \longrightarrow 0$$

*in $\mathcal{A}$.*

*Proof.* Lemma 4.4 assures that for an $\mathcal{E}$-proper chain complex $C$, its homology may be computed as expressed in the following diagram.

$$\begin{array}{ccccccccc}
\cdots \to C_{n+2} & \xrightarrow{d_{n+2}} & C_{n+1} & \xrightarrow{d_{n+1}} & & C_n & \xrightarrow{d_n} & C_{n-1} \to \cdots \\
 & & {\scriptstyle \mathrm{Coker}\, d_{n+2}}\downarrow & & & \uparrow{\scriptstyle \mathrm{Ker}\, d_n} & & \\
\mathrm{H}_{n+1}C & \xrightarrow[\mathrm{Ker}\, \overline{d_{n+1}}]{} & \mathrm{Cok}[d_{n+2}] & \xrightarrow[\overline{d_{n+1}}]{} & & \mathrm{K}[d_n] & \xrightarrow[\mathrm{Coker}\, \overline{d_{n+1}}]{} & \mathrm{H}_n C
\end{array}$$

Note that the uniquely induced morphism $\overline{d_{n+1}}$ is $\mathcal{E}$-proper by Lemma 2.5. As in the abelian case, one uses the Snake Lemma (here in its relative version [12, Lemma 4.2]) twice to obtain an $\mathcal{E}$-exact sequence

$$\mathrm{H}_{n+1}C \longrightarrow \mathrm{H}_{n+1}D \longrightarrow \mathrm{H}_{n+1}E \longrightarrow \mathrm{H}_n C \longrightarrow \mathrm{H}_n D \longrightarrow \mathrm{H}_n E$$

in $\mathcal{A}$ for every $n \geq 0$. □

**Relative semi-abelian categories.** In order for homology of simplicial objects to make sense, we have to go to the stronger setting of *relative semi-abelian categories* also introduced by T. Janelidze [14], so that the Moore complex of a simplicial object is $\mathcal{E}$-proper. The definition of these categories is usually given via some properties of relations, but we give an alternative equivalent definition as we do not need relations for anything else.

**Definition 4.6.** [14] A **relative semi-abelian category** is a relative homological category $(\mathcal{A}, \mathcal{E})$ where $\mathcal{A}$ has binary coproducts and the pushout of an extension by an extension exists and is a double extension.

This definition is equivalent to the one in [14] via [14, Theorem 3.3] and the arguments for [3, Theorem 5.7].

Via Theorem 3.5 in [14], Axiom (F) implies:

**Lemma 4.7.** *If a morphism $f$ in a relative semi-abelian category $(\mathcal{A}, \mathcal{E})$ factors as $f = e \circ m$ with $m$ an $\mathcal{E}$-normal monomorphism and $e \in \mathcal{E}$, then it also factors (essentially uniquely) as $f = m' \circ e'$ with $m'$ an $\mathcal{E}$-normal monomorphism and $e' \in \mathcal{E}$.* □

We also need a property of $\mathcal{E}$-normal monomorphisms which holds in relative semi-abelian categories.

**Lemma 4.8.** *In a relative semi-abelian category, finite intersections of $\mathcal{E}$-normal monomorphisms are $\mathcal{E}$-normal monomorphisms.*

*Proof.* This follows from the part of the definition which says that a pushout of an extension by an extension exists and is a double extension. □

**Homology of $\mathcal{E}$-simplicial objects.** In order to define homology of $\mathcal{E}$-simplicial objects, we must first show that we can in fact apply the definition of homology above to the Moore complex of any $\mathcal{E}$-simplicial object. We use the following relative version of [8, Theorem 3.6].

**Lemma 4.9.** *If $\mathbb{A}$ is an augmented $\mathcal{E}$-semi-simplicial object in a relative semi-abelian category $(\mathcal{A}, \mathcal{E})$ then the Moore complex $\mathrm{N}\mathbb{A}$ of $\mathbb{A}$ is $\mathcal{E}$-proper.*

*Proof.* Since $\partial_{n+1}$ is in $\mathcal{E}$, the composite morphism $d$ in the diagram

$$\begin{array}{ccc}
\mathrm{N}_{n+1}\mathbb{A} & \xrightarrow{\bigcap_i \mathrm{Ker}\, \partial_i} & A_{n+1} \\
{\scriptstyle d_{n+1}}\downarrow & \overset{d}{\searrow} & \downarrow{\scriptstyle \partial_{n+1}} \\
\mathrm{N}_n\mathbb{A} & \xrightarrow[\bigcap_i \mathrm{Ker}\, \partial_i]{} & A_n
\end{array}$$

factors as an extension $e$ followed by a monomorphism $m$. Lemma 4.8 states that the top map $\bigcap_i \operatorname{Ker} \partial_i \colon \mathrm{N}_{n+1}\mathbb{A} \to A_{n+1}$ is an $\mathcal{E}$-normal monomorphism, so that the monomorphism $m$ is $\mathcal{E}$-normal by Lemma 4.7. Lemma 2.5 now implies that the factorisation $l$ of $m$ over $\mathrm{N}_n\mathbb{A}$ is also an $\mathcal{E}$-normal monomorphism. □

Thus now we can define:

**Definition 4.10** (Homology of $\mathcal{E}$-semi-simplicial objects)**.** Let $\mathbb{A}$ be an (augmented) $\mathcal{E}$-semi-simplicial object in a relative semi-abelian category $(\mathcal{A}, \mathcal{E})$. The **$n$th homology object of** $\mathbb{A}$ is defined by

$$\mathrm{H}_n\mathbb{A} = \mathrm{H}_n\mathrm{N}\mathbb{A}$$

for $n \geq 1$, and

$$\mathrm{H}_0\mathbb{A} = \operatorname{Cok}[d_1 \colon N_1\mathbb{A} \to A_0]$$

for $n = 0$.

Via Proposition 4.3, this allows us to rephrase Theorem 3.9 in terms of homology.

**Theorem 4.11.** *In a relative semi-abelian category $(\mathcal{A}, \mathcal{E})$, an augmented $\mathcal{E}$-(quasi)-simplicial object $\mathbb{A}$ in $\mathcal{A}$ is an $\mathcal{E}$-resolution if and only if $\mathrm{H}_n\mathbb{A} = 0$ for all $n \geq 1$ and $\mathrm{H}_0\mathbb{A} = A_{-1}$.* □

We then also have a long exact homology sequence for simplicial objects:

**Theorem 4.12** (Long exact homology sequence, simplicial case)**.** *In a relative semi-abelian category $(\mathcal{A}, \mathcal{E})$, any short $\mathcal{E}$-exact sequence of $\mathcal{E}$-quasi-simplicial objects*

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathbb{A} \longrightarrow \mathbb{B} \longrightarrow 0$$

*gives a long $\mathcal{E}$-exact homology sequence*

$$\cdots \longrightarrow \mathrm{H}_{n+1}\mathbb{K} \longrightarrow \mathrm{H}_{n+1}\mathbb{A} \longrightarrow \mathrm{H}_{n+1}\mathbb{B} \longrightarrow \mathrm{H}_n\mathbb{K} \longrightarrow \cdots \longrightarrow \mathrm{H}_0\mathbb{B} \longrightarrow 0$$

*in $\mathcal{A}$.*

*Proof.* This follows from Proposition 4.5 via Proposition 3.8. □

## 5. Homology with coefficients in a relative semi-abelian category

When dealing with homology of objects (rather than complexes), a very important aspect is its independence of a chosen projective resolution. In this section we show that any two projective $\mathcal{E}$-resolutions of an object are simplicially homotopic and thus give rise to the same homology when sent into a relative semi-abelian category by a functor preserving extensions.

**The relative Kan property.** We will need a particular property of simplicial objects in relative semi-abelian categories which comes from relative Mal'tsev categories [6]. We first define this, a relative Kan property, which is a slight adaptation of the Kan property in a regular category given in [3].

**Definition 5.1.** Let $\mathbb{A}$ be a semi-simplicial object and consider $n \geq 2$ and $0 \leq k \leq n$. The **object of $(n, k)$-horns in** $\mathbb{A}$ is an object $A(n, k)$ together with arrows $a_i \colon A(n, k) \to A_{n-1}$ for $i \in \{0, \ldots, n\} \setminus \{k\}$ satisfying

$$\partial_i \circ a_j = \partial_{j-1} \circ a_i \text{ for all } i < j \text{ with } i,\, j \neq k$$

which is universal with respect to this property. We also define $A(1, 0) = A(1, 1) = A_0$.

A semi-simplicial object is **$\mathcal{E}$-Kan** when all objects $A(n, k)$ exist and all comparison morphisms $A_n \to A(n, k)$ are in $\mathcal{E}$. (In particular, the comparison morphisms to the $(1, k)$-horns are $\partial_0 \colon A_1 \to A(1, 0)$ and $\partial_1 \colon A_1 \to A(1, 1)$.)

Notice again that if $\mathcal{A}$ has all pullbacks, then all horn objects exist.

We now use a result from [6] which is stated there in a slightly different context, but it is explained in Section 4 in the same paper that it also applies to a context of which relative homological categories (and thus relative semi-abelian categories) form an example.

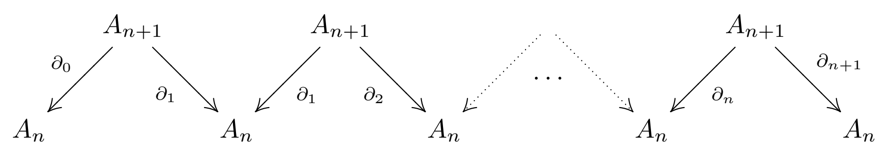
**Theorem 5.2.** *Let $(\mathcal{A}, \mathcal{E})$ be a relative homological category. Then every $\mathcal{E}$-simplicial object is $\mathcal{E}$-Kan.*

*Proof.* This follows from Theorem 3.11 in [6] together with the explanations in Section 4 of the same paper, which ensure that relative homological categories form an example of the situation considered there. In fact, relative homological categories are *relatively Mal'tsev* in the sense of Definition 4.8 in [6]. $\square$

**Simplicial homotopy.** We first prove an important property of semi-simplicial homotopy in a relative homological context: the kernel of a cocylinder projection $\epsilon_0\colon \mathbb{A}^I \to \mathbb{A}$ is always an $\mathcal{E}$-resolution.

**Remark 5.3.** In the proof of Proposition 5.5, we have to show that, in a particular simplicial object, the comparison morphisms $c\colon C_n \to K_n\mathbb{C}$ to the simplicial kernels are in $\mathcal{E}$. This can be done by showing that, for any $b\colon B \to K_n\mathbb{C}$, there exists an extension $p\colon Y \to B$ and a morphism $y\colon Y \to A_n$ such that $b\circ p = c\circ y$. This means using generalised elements and *enlargement of domain* as for example in [3]. However, to make the proof easier to read, we just write it down as if we were using actual elements and suppress the enlargement of domain. Similarly, we will use element notation to mean generalised elements in the following definition.

**Definition 5.4** (Cocylinder object)**.** Let $(\mathcal{A}, \mathcal{E})$ satisfy (E1)–(E3). Let $\mathbb{A}$ be an $\mathcal{E}$-semi-simplicial object in $\mathcal{A}$. Put $A_0^I = A_1$ and, for $n > 0$, let $A_n^I$ be the limit of the zigzag



in $\mathcal{A}$. The faces $\partial_i^I\colon A_n^I \to A_{n-1}^I$ defined by

$$\partial_i^I(a_0, \dots, a_n) = (\partial_{i+1}a_0, \dots, \partial_{i+1}a_{i-1}, \partial_i a_{i+1}, \dots, \partial_i a_n)$$

for $n \geq 1$ determine an $\mathcal{E}$-semi-simplicial object $\mathbb{A}^I$.

Furthermore, $\epsilon_0$, $\epsilon_1\colon \mathbb{A}^I \to \mathbb{A}$ denote the semi-simplicial morphisms defined by

$$(\epsilon_0)_n(a_0, \dots, a_n) = \partial_0 a_0, \qquad (\epsilon_1)_n(a_0, \dots, a_n) = \partial_{n+1}a_n$$

for all $n \geq 0$. Note that degreewise they are extensions.

When $\mathbb{A}$ is an $\mathcal{E}$-quasi-simplicial object, the projections $\epsilon_0$ and $\epsilon_1$ admit a common splitting $s\colon \mathbb{A} \to \mathbb{A}^I$ (in the category of $\mathcal{E}$-semi-simplicial objects) defined by $s_n = \langle \sigma_0, \dots, \sigma_n \rangle\colon A_n \to A_n^I$ for all $n \geq 0$.

When $\mathbb{A}$ is an augmented $\mathcal{E}$-quasi-simplicial object, we also put $A_{-1}^I = A_{-1}$ and $(\epsilon_0)_{-1} = (\epsilon_1)_{-1} = s_{-1} = 1_{A_{-1}}$.

It is clear that two semi-simplicial morphisms $f$, $g\colon \mathbb{A} \to \mathbb{B}$ are homotopic if and only if the morphism $\langle f, g \rangle\colon \mathbb{A} \to \mathbb{B} \times \mathbb{B}$ factors over $\langle \epsilon_0, \epsilon_1 \rangle\colon \mathbb{B}^I \to \mathbb{B} \times \mathbb{B}$. (If $\mathbb{A}$ and $\mathbb{B}$ are augmented, under both conditions $f$ and $g$ coincide at level $-1$.)

**Proposition 5.5.** *Suppose that $(\mathcal{A}, \mathcal{E})$ is a relative homological category. Let $\mathbb{A}$ be an augmented $\mathcal{E}$-quasi-simplicial object in $\mathcal{A}$. Then the kernel of $\epsilon_0\colon \mathbb{A}^I \to \mathbb{A}$ is an $\mathcal{E}$-resolution.*

*Proof.* Notice that $\mathrm{K}[(\epsilon_0)_{-1}] = 0$, and $\mathrm{K}[(\epsilon_0)_0] \to 0$ is in $\mathcal{E}$ as the kernel of the double extension $(\partial_0, 1_{A_{-1}})\colon \partial_0\partial_0 \to \partial_0$. This means that $\mathrm{K}[\epsilon_0]$ is an $\mathcal{E}$-resolution at level 0.

Now let $n \geq 1$. As $(\mathcal{A}, \mathcal{E})$ is relatively homological, the $\mathcal{E}$-quasi-simplicial object $\mathbb{A}$ is $\mathcal{E}$-Kan by Theorem 5.2. As remarked above, we write the proof using elements instead of generalised elements. Any $(n+1)$-tuple $(a_0, \dots, a_n) \in \mathrm{K}[(\epsilon_0)_n] \subseteq A_n^I$ determines an element of $K_n\mathrm{K}[\epsilon_0] \subseteq K_n\mathbb{A}^I$ which may be expressed in a matrix as follows.

$$\begin{matrix} \partial_0 a_1 & \partial_0 a_2 & \partial_0 a_3 & \dots & \partial_0 a_n \\ \partial_2 a_0 & \partial_1 a_2 & \partial_1 a_3 & \dots & \partial_1 a_n \\ \partial_3 a_0 & \partial_3 a_1 & \partial_2 a_3 & \dots & \partial_2 a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \partial_{n+1} a_0 & \partial_{n+1} a_1 & \partial_{n+1} a_2 & \dots & \partial_{n+1} a_{n-1} \end{matrix}$$

The rows of this matrix are elements of $\mathrm{K}[(\epsilon_0)_{n-1}] \subseteq A^I_{n-1}$. Given an arbitrary element of $K_n\mathrm{K}[\epsilon_0]$, i.e., a matrix

$$\begin{matrix} b_{0,1} & b_{0,2} & b_{0,3} & \dots & b_{0,n} \\ b_{2,0} & b_{1,2} & b_{1,3} & \dots & b_{1,n} \\ b_{3,0} & b_{3,1} & b_{2,3} & \dots & b_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n+1,0} & b_{n+1,1} & b_{n+1,2} & \dots & b_{n+1,n-1} \end{matrix}$$

satisfying certain properties, we must find an $(n+1)$-tuple $(a_0, \dots, a_n)$ in $\mathrm{K}[(\epsilon_0)_n] \subseteq A^I_n$ such that the two matrices above are equal. We construct this required $(n+1)$-tuple by induction using the Kan property.

We start by constructing $a_0$. Note that such an element of $\mathrm{K}[(\epsilon_0)_n]$ necessarily satisfies $\partial_0 a_0 = 0$ and $\partial_i a_0 = b_{i,0}$ for all $2 \leq i \leq n+1$. These conditions determine an $(n+1,1)$-horn in $\mathbb{A}$, which by the Kan property gives rise to the needed element $a_0$.

Now suppose that, for some $0 \leq k-1 \leq n-1$, the elements $a_0, \dots, a_{k-1}$ have been constructed. Then the next element $a_k$ may be constructed as follows. It must satisfy $\partial_i a_k = b_{i,k}$ for all $0 \leq i \leq n$ such that $i \neq k$, $k+1$ as well as the equality $\partial_k a_k = \partial_k a_{k-1}$ given by the induction hypothesis. These equations determine an $(n+1, k+1)$-horn in $\mathbb{A}$, which by the Kan property induces the needed element $a_k$. □

Notice that, when $\mathbb{A}$ is not augmented, we can still say that $\mathrm{K}[\epsilon_0]$ is $\mathcal{E}$-exact for $n \geq 2$, but the simplicial kernel $K_1\mathrm{K}[\epsilon_0]$ does not make sense for a non-augmented (quasi)-simplicial object.

**Corollary 5.6.** *Let $\mathbb{B}$ be an $\mathcal{E}$-quasi-simplicial object in a relative semi-abelian category $(\mathcal{A}, \mathcal{E})$. If $f$ and $g\colon \mathbb{A} \to \mathbb{B}$ are homotopic $\mathcal{E}$-semi-simplicial morphisms, then for any $n \geq 0$ the maps $\mathrm{H}_n f$ and $\mathrm{H}_n g\colon \mathrm{H}_n\mathbb{A} \to \mathrm{H}_n\mathbb{B}$ are equal.*

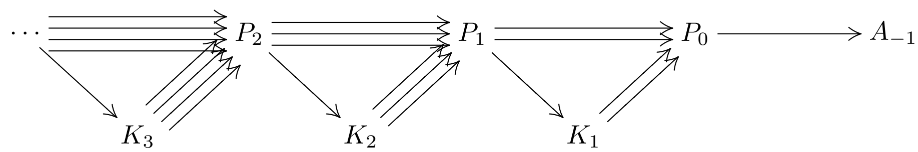
*Proof.* It suffices to show that $\mathrm{H}_n\epsilon_0 = \mathrm{H}_n\epsilon_1\colon \mathrm{H}_n\mathbb{B}^I \to \mathrm{H}_n\mathbb{B}$. For $n = 0$ this is clear. This implies that we can augment $\mathbb{A}$ and $\mathbb{B}$ by $\mathrm{H}_0\mathbb{A}$ and $\mathrm{H}_0\mathbb{B}$ respectively, giving $\mathrm{H}_0 f = \mathrm{H}_0 g$ as augmentation for $f$ and $g$. For $n \geq 1$ we can now use Proposition 5.5 together with Theorem 4.11: indeed, being an $\mathcal{E}$-resolution, the homology of $\mathrm{K}[\epsilon_0]$ is trivial for $n \geq 1$; via the long exact homology sequence (Theorem 4.12) and the fact that $\epsilon_0$ is a split epimorphism of $\mathcal{E}$-semi-simplicial objects, this implies that all $\mathrm{H}_n\epsilon_0$ are isomorphisms. Since both $\mathrm{H}_n\epsilon_0$ and $\mathrm{H}_n\epsilon_1$ are split by the same morphism $\mathrm{H}_n s$, they are equal. □

**Homology of objects.** The homology of an object $A$ of $\mathcal{A}$ depends on a functor $I\colon \mathcal{A} \to \mathcal{B}$. For this to make sense, we need very few conditions on the pair $(\mathcal{A}, \mathcal{E})$, but we need the pair $(\mathcal{B}, \mathcal{F})$, where the homology is actually calculated, to be relatively semi-abelian. Thus we now assume that $(\mathcal{A}, \mathcal{E})$ satisfies (E1)–(E3), and that $I$ is a functor to a relative semi-abelian category $(\mathcal{B}, \mathcal{F})$ sending $\mathcal{E}$ into $\mathcal{F}$.

**Definition 5.7.** Let $(\mathcal{A}, \mathcal{E})$ satisfy (E1)–(E3). A **projective $\mathcal{E}$-resolution** of an object $A$ of $\mathcal{A}$ is a quasi-simplicial $\mathcal{E}$-resolution $\mathbb{A}$ of $A$ where for all $n \geq 0$ the objects $A_n$ are projective with respect to the class $\mathcal{E}$.

Notice that an $\mathcal{E}$-resolution, and thus a projective $\mathcal{E}$-resolution, is automatically an $\mathcal{E}$-semi-simplicial object.

**Remark 5.8** (Tierney-Vogel construction of projective resolutions)**.** If $(\mathcal{A}, \mathcal{E})$ is a pair such that $\mathcal{A}$ has enough $\mathcal{E}$-projectives, we can construct a projective $\mathcal{E}$-resolution as follows [17]. First we choose an $\mathcal{E}$-projective object $P_0$ and a morphism $\partial_0 \colon P_0 \to A$ in $\mathcal{E}$, which we may call a **(projective) cover of** $A$. Then the simplicial kernel $K_1$ of $\partial_0$ (the kernel pair) can be covered by another $\mathcal{E}$-projective object $P_1$, resulting in the composites $\partial_0$ and $\partial_1 \colon P_1 \to P_0$. Thus we successively take the simplicial kernel and cover it by an $\mathcal{E}$-projective object to obtain the quasi-simplicial object $\mathbb{P}$




The degeneracies are induced inductively by the limit properties of the simplicial kernels and lift to the $P_n$ as these are projective. This augmented quasi-simplicial object is an $\mathcal{E}$-resolution by construction.

This proves that, when $\mathcal{A}$ has enough $\mathcal{E}$-projectives, projective $\mathcal{E}$-resolutions of an object always exist, but of course they are not unique. However, as we shall now show, two such resolutions give rise to the same homology of an object $A$. For this we use the Comparison Theorem from [17], which we state here using our own terminology.

**Theorem 5.9** ([17, Theorem 2.4])**.** *Let $(\mathcal{A}, \mathcal{E})$ satisfy (E1)–(E3). Given two objects $A$ and $B$ in $\mathcal{A}$, let $\mathbb{A}$ be a semi-simplicial object over $A$ with all $A_n$, for $n \geq 0$, being $\mathcal{E}$-projectives, and let $\mathbb{B}$ be an $\mathcal{E}$-resolution of $B$. Then any morphism $f \colon A \to B$ can be extended to a semi-simplicial morphism $f \colon \mathbb{A} \to \mathbb{B}$, and any two such extensions are simplicially homotopic.* □

As simplicial homotopies are preserved by any functor, this immediately gives the following

**Theorem 5.10.** *Let $(\mathcal{A}, \mathcal{E})$ satisfy (E1)–(E3) and let $I \colon \mathcal{A} \to \mathcal{B}$ be a functor to a relative semi-abelian category $(\mathcal{B}, \mathcal{F})$ sending $\mathcal{E}$ into $\mathcal{F}$. Given two projective $\mathcal{E}$-resolutions $\mathbb{A}$ and $\mathbb{B}$ of an object $A$ in $\mathcal{A}$, the objects $\mathrm{H}_n I\mathbb{A}$ and $\mathrm{H}_n I\mathbb{B}$ are isomorphic for any $n \geq 0$.*

*Proof.* Using the Comparison Theorem 5.9 on the identity $1_A \colon A \to A$, we see that the resolutions $\mathbb{A}$ and $\mathbb{B}$ are homotopy equivalent. As the functor $I$ preserves this homotopy equivalence, the $\mathcal{F}$-quasi-simplicial objects $I\mathbb{A}$ and $I\mathbb{B}$ are also homotopy equivalent and thus Corollary 5.6, used in the relative semi-abelian category $(\mathcal{B}, \mathcal{F})$, implies that $\mathrm{H}_n I\mathbb{A} \cong \mathrm{H}_n I\mathbb{B}$ for all $n \geq 0$. □

**Remark 5.11.** The classes $\mathcal{E}$ and $\mathcal{F}$ play slightly different roles in this result: $\mathcal{E}$ in $\mathcal{A}$ provides a projective class which allows us to form projective resolutions, and so does not need to satisfy many axioms. $\mathcal{F}$ in $\mathcal{B}$ makes $(\mathcal{B}, \mathcal{F})$ relative semi-abelian and so makes sure the homology is well-defined. To be able to use the theory developed in this paper, we need the (quasi)-simplicial objects to be $\mathcal{F}$-(quasi)-simplicial, so for this reason $I$ does have to preserve extensions even though they play such different roles. Notice that even if $I$ did not preserve extensions, it would still preserve homotopies.

We can now define homology with coefficients in $I$.

**Definition 5.12** (Homology of an object)**.** Let $(\mathcal{A}, \mathcal{E})$ satisfy (E1)–(E3) and let $I \colon \mathcal{A} \to \mathcal{B}$ be a functor into a relative semi-abelian category $(\mathcal{B}, \mathcal{F})$ sending $\mathcal{E}$ into $\mathcal{F}$. Given an object $A$ in $\mathcal{A}$, let $\mathbb{A}$ be a projective $\mathcal{E}$-resolution of $A$. For any $n \geq 0$, the $(n+1)$**st homology object of** $A$ **with coefficients in** $I$ is the object

$$\mathrm{H}_{n+1}(A, I) = \mathrm{H}_n I\mathbb{A}.$$

When $\mathcal{A}$ has enough $\mathcal{E}$-projectives, this defines a functor

$$\mathrm{H}_{n+1}(-, I) \colon \mathcal{A} \to \mathcal{B}.$$

The dimension shift stems from the classical numbering of homology of groups and similar algebraic objects.

Apart from enabling this definition in a relative context, our proofs of the results leading up to this definition are also a good way to view the corresponding absolute results. We have endeavoured to present them as clearly as possible, also giving explicit explanations for results which so far have been always used but never completely and explicitly written down.

**Acknowledgements.** I would like to thank Tomas Everaert and Tim Van der Linden for their invaluable help with this paper, which is a direct by-product of our collaboration on [6].

## References

1. D. Bourn, $3 \times 3$ *Lemma and protomodularity*, J. Algebra **236** (2001), 778–795.
2. D. Bourn, *The denormalized* $3 \times 3$ *lemma*, J. Pure Appl. Algebra **177** (2003), 113–129.
3. A. Carboni, G. M. Kelly, and M. C. Pedicchio, *Some remarks on Maltsev and Goursat categories*, Appl. Categ. Structures **1** (1993), 385–421.
4. G. Donadze, N. Inassaridze, and T. Porter, *n-Fold Čech derived functors and generalised Hopf type formulas*, K-Theory **35** (2005), no. 3–4, 341–373.
5. T. Everaert, *Higher central extensions and Hopf formulae*, J. Algebra **324** (2010), 1771–1789.
6. T. Everaert, J. Goedecke, and T. Van der Linden, *Resolutions, higher extensions and the relative Mal'tsev axiom*, Pré-Publicações DMUC **10-49** (2010), 1–44, submitted.
7. T. Everaert, M. Gran, and T. Van der Linden, *Higher Hopf formulae for homology via Galois Theory*, Adv. Math. **217** (2008), no. 5, 2231–2267.
8. T. Everaert and T. Van der Linden, *Baer invariants in semi-abelian categories II: Homology*, Theory Appl. Categ. **12** (2004), no. 4, 195–224.
9. S. A. Huq, *Commutator, nilpotency and solvability in categories*, Quart. J. Math. Oxford **19** (1968), no. 2, 363–389.
10. G. Janelidze, *Pure Galois theory in categories*, J. Algebra **132** (1990), no. 2, 270–286.
11. G. Janelidze, *What is a double central extension? (The question was asked by Ronald Brown)*, Cah. Topol. Géom. Differ. Catég. **XXXII** (1991), no. 3, 191–201.
12. T. Janelidze, *Relative homological categories*, J. Homotopy Relat. Struct. **1** (2006), no. 1, 185–194.
13. T. Janelidze, *Foundation of relative non-abelian homological algebra*, Ph.D. thesis, University of Cape Town, 2009.
14. T. Janelidze, *Relative semi-abelian categories*, Appl. Categ. Structures **17** (2009), 373–386.
15. S. Mac Lane, *Categories for the working mathematician*, second ed., Grad. Texts in Math., vol. 5, Springer, 1998.
16. R. Paré, *Simply connected limits*, Canad. J. Math. **42** (1990), 731–746.
17. M. Tierney and W. Vogel, *Simplicial resolutions and derived functors*, Math. Z. **111** (1969), no. 1, 1–14.
18. T. Van der Linden, *Simplicial homotopy in semi-abelian categories*, J. K-Theory **4** (2009), no. 2, 379–390.

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, England

*E-mail address*: julia.goedecke@cantab.net