Groups Ia Practice Sheet A

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These questions are not supposed to form the work for one of the regular 4 groups supervisions, but instead they give you opportunities to practise getting used to axioms and definitions in your own time. If you find this useful, try to make similar questions for yourself on later material of the course.

Examples of groups

1. Show that $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ (rational numbers without zero) is a group under multiplication. [This means that you should use standard multiplication as the operation for the group.] Is $\mathbb{R}^+$ (positive real numbers) a group under multiplication?

2. Show that $\mathbb{R}^3$ with (componentwise) addition (you might know it as vector addition) is a group. What is the inverse to a general vector (x, y, z)?

3. Do the natural numbers $\mathbb{N}$ (with or without zero, choose which you prefer) form a group under addition? If yes, show carefully that all axioms hold. If no, show which axioms do not hold.

4. Is $\mathbb{Q}$ a group under multiplication? Again prove your answer carefully.

5. Show that $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ does not form a group under division. Which axioms do not hold? [You should in particular pay attention to the identity axiom.]

6. On the integers $\mathbb{Z}$ define the operation $n * m = n + m + 1$. Show that $\mathbb{Z}$ forms a group under this operation. What is the identity element? Show that $-(n + 2)$ is the inverse to n. Is the group abelian?

7. Show that composition of functions is always an associative operation. What extra properties might you need to get a group with composition as the group operation?

8. * *This is a more involved question, but still very interesting to try.*

   Let X be a set. The **powerset** $\mathcal{P}(X)$ is the set of all subsets of X:

   $$\mathcal{P}(X) = \{A \mid A \subseteq X\}$$

   Does $\mathcal{P}(X)$ form a group under the operation $\cap$ (intersection)? Does $\mathcal{P}(X)$ form a group under the operation $\cup$ (union)?

   The **symmetric difference** of two subsets $A, B \subseteq X$ is the set of elements which are in *exactly one* of A and B:

   $$A \triangle B = (A \cup B) \setminus (A \cap B).$$

   Show that $\mathcal{P}(X)$ forms a group under $\triangle$. What is the identity element? For now, assume that symmetric difference is an associative operation. You will meet a nice way of proving it later in Numbers and Sets.