Groups Example Sheet 4

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1. Let G be the set of all 3×3 matrices of the form
$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$
with $x, y, z \in \mathbb{R}$. Show that G is a subgroup of the group of invertible real matrices under multiplication. Let H be the subset of G given by those matrices with $x = z = 0$. Show that H is a normal subgroup of G and identify G/H. (This G is called the *Heisenberg group.*)

2. Take the Heisenberg group as above, but this time with entries in $\mathbb{Z}_3$. Show that every non-identity element of this group has order 3, but the group is not isomorphic to $C_3 \times C_3 \times C_3$.

3. Recall that the *centre* of a group G consists of all those elements of G that commute with all the elements of G. Show that the centre Z of the general linear group $GL_2(\mathbb{C})$ consists of all non-zero scalar matrices. Identify the centre of the special linear group $SL_2(\mathbb{C})$.

4. Consider the set of matrices of the form $\left(\begin{smallmatrix} 0 & 0 \\ 0 & t \end{smallmatrix}\right)$ for $t \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$. Show that these form a group under matrix multiplication. More generally, show that if a set of matrices forms a group under multiplication, then either all matrices in the set have non-zero determinant, or all have zero determinant.

5. Let G be the set of all 3×3 real matrices of determinant 1 of the form
$$\begin{pmatrix} a & 0 & 0 \\ b & x & y \\ c & z & w \end{pmatrix}.$$
Verify that G is a group. Find a homomorphism from G onto the group $GL_2(\mathbb{R})$ of all non-singular 2×2 real matrices, and find its kernel.

6. Let K be a normal subgroup of order 2 in the group G. Show that K lies in the centre of G; that is, show $kg = gk$ for all $k \in K$ and $g \in G$.

   Exhibit a surjective homomorphism of the orthogonal group $O(3)$ onto C_2 and another onto the special orthogonal group $SO(3)$.

7. Consider the Möbius maps $f(z) = e^{2\pi i/n} z$ and $g(z) = 1/z$. Show that the subgroup $G = \langle f, g \rangle$ of the Möbius group $\mathcal{M}$ is a dihedral group of order $2n$.

8. Let $g(z) = (z + 1)/(z - 1)$. By considering the points $g(0)$, $g(\infty)$, $g(1)$ and $g(i)$, find the image of the real axis $\mathbb{R}$ and of the imaginary axis $\mathbb{I}$ under g. What is $g(\Sigma)$, where Σ is the first quadrant in $\mathbb{C}$?

9. What is the order of the Möbius map $f(z) = iz$? If h is any Möbius map, find the order of hfh^{-1} and its fixed points. Use this to construct a Möbius map of order four that fixes 1 and -1.

10. Let G be the group of Möbius transformations which map the set $\{0, 1, \infty\}$ onto itself. Find all the elements in G. To which standard group is G isomorphic? Justify your answer.

    Find the group of Möbius transformations which map the set $\{0, 2, \infty\}$ onto itself. [Try to do as little calculation as possible.]

11. Let G be as in the previous question. Show that, given $\sigma \in S_4$, there exists $f_\sigma \in G$ for which, whenever z_1, z_2, z_3 and z_4 are four distinct points in $\mathbb{C}_\infty$, we have $f_\sigma([z_1, z_2, z_3, z_4]) = [z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}]$. [You may want to start with σ a transposition in S_4.]

    Show that the map $\sigma \longmapsto f_{\sigma^{-1}}$ from S_4 to G gives a homomorphism from S_4 onto S_3. Find its kernel.

12. Let G be the general linear group $\mathrm{GL}_2(5)$ of invertible 2×2 matrices over the field $\mathbb{F}_5$ of integers modulo 5, so that the arithmetic in G is modulo 5. Let H be the subgroup $\mathrm{SL}_2(5)$ of G consisting of matrices of determinant 1. Show that G has order 480. By considering a suitable homomorphism from G to another group, deduce that H has order 120.

    Prove that $-I$ is the only element of H of order 2, and deduce that H has no subgroup isomorphic to A_5 .

    Find a subgroup of H isomorphic to Q_8, and an element of order 3 normalising it in H. Deduce that H has a subgroup of index 5, and obtain a homomorphism from H to S_5.

    * Deduce that $\mathrm{SL}_2(5)/\{\pm I\}$ is isomorphic to the alternating group A_5.