Groups Example Sheet 3

Michaelmas 2015 Julia Goedecke
Please send comments and corrections to jg352.

1. Let D_{2n} be the group of symmetries of a regular n-gon. Show that **any** subgroup K of rotations is normal in D_{2n}, and identify the quotient D_{2n}/K. (Identify means: what standard group is it isomorphic to?)

2. Show that D_{2n} has two conjugacy classes of reflections if n is even, but only one if n is odd.

3. Let Q be a plane quadrilateral. Show that its group $G(Q)$ of symmetries has order at most 8. For each n in the set $\{1, 2, \ldots, 8\}$, either give an example of a quadrilateral Q with $G(Q)$ of order n, or show that no such quadrilateral can exist.

4. List all the subgroups of the dihedral group D_8, and indicate which pairs of subgroups are isomorphic.
   Repeat for the quaternion group Q_8.

5. Find the conjugacy classes of D_8 and their sizes. Show that the centre Z of the group has order 2, and identify the quotient group D_8/Z of order 4.
   Repeat with the quaternion group Q_8.

6. What is the group of all rotational symmetries of a Toblerone box, a solid triangular prism with an equilateral triangle as a cross-section, with ends orthogonal to the longitudinal axis of the prism? And the group of all symmetries?

7. Suppose that the group G acts on the set X. Let $x \in X$, let $y = g(x)$ for some $g \in G$. Show that the stabiliser G_y equals the conjugate gG_xg^{-1} of the stabiliser G_x.

8. Let G be a finite group and let X be the set of all subgroups of G. Show that G acts on X by $g : H \longmapsto gHg^{-1}$ for $g \in G$ and $H \in X$, where $gHg^{-1} = \{ghg^{-1} : h \in H\}$. Show that the orbit containing H in this action of G has size at most $|G|/|H|$. If H is a proper subgroup of G, show that there exists an element of G which is contained in no conjugate gHg^{-1} of H in G.

9. Let G be a finite group of prime power order p^a, with $a > 0$. By considering the conjugation action of G, show that the centre Z of G is non-trivial.
   Show that any group of order p^2 is abelian, and that there are up to isomorphism just two groups of that order for each prime p.

10. Find the conjugacy classes of elements in the alternating group A_5, and determine their sizes. Show that A_5 has no non-trivial normal subgroups (so A_5 is a *simple* group).
    Show that if H is a proper subgroup of index n in A_5 then $n > 4$. [Consider the left coset action of A_5 on the set of left cosets of H in A_5.]