Groups Example Sheet 2

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Please send comments and corrections to jg352.

The questions are not necessarily in order of difficulty — more in order of lecture material.

0. If you did not do Questions 12 and 13 on Sheet 1 (because we had not covered all the material in lectures), then do them now.

1. Let G be a subgroup of the symmetric group S_n. Show that if G contains any odd permutations then precisely half of the elements of G are odd.

2. A finite group G is generated by a set T of elements of G if each element of G can be written as a finite product (possibly with repetitions) of (positive and negative) powers of elements of T. Show that the symmetric group S_n is generated by each of the following sets of permutations:

   (a) the set $\{(j, j+1) : 1 \leq j < n\}$;

   (b) the set $\{(1, k) : 1 < k \leq n\}$;

   (c) the set $\{(1, 2), (12 \ldots n)\}$ consisting of a transposition and an n-cycle.

   Show also that A_n is generated by the set of 3-cycles.

3. (a) Show that the symmetric group S_4 has a subgroup of order d for each divisor d of 24, and find two non-isomorphic subgroups of order 4.

   (b) Show that the alternating group A_4 has a subgroup of each order up to 4, but there is no subgroup of order 6.

4. Let H be a subgroup of the group G. Find a (natural) bijection between the set of all left cosets and the set of all right cosets of H in G.

5. (a) Let H be a subgroup of the (possibly infinite) group G, let K be a subgroup of H. We are given the list of disjoint cosets of H in G as $\{g_i H \mid i \in I\}$ and the disjoint cosets of K in H are $\{h_j K \mid j \in J\}$, for I and J some sets. Give a list of all disjoint cosets of K in G, and justify your answer. Deduce that when G is finite, then the index $|G : K|$ equals the product $|G : H||H : K|$.

   (b) Let G be an infinite group, and let H and K be subgroups of G of finite index. Show that $H \cap K$ has finite index.

6. Show that if a group G contains an element of order six, and an element of order ten, then G has order at least 30.

7. Show that the set $\{1, 3, 5, 7\}$ with multiplication modulo 8 is a group. Is this group isomorphic to C_4 or $C_2 \times C_2$? Justify your answer.

8. (a) Show that the set of functions on $\mathbb{R}$ of the form $f(x) = ax + b$, where a and b are real numbers and $a \neq 0$, forms a group under composition of functions. Is this group abelian? [This is another example of an infinite transformation group.]

   (b) Consider the corresponding set of functions on $\mathbb{Z}_n$, the integers modulo n, i.e. functions of form $f(x) = ax + b$ with $a, b \in \mathbb{Z}_n$ and $a \neq 0$. For which n does this set form a group? Are any of these groups abelian?

9. Consider the set of matrices of the form $\left(\begin{smallmatrix} 0 & 0 \\ 0 & t \end{smallmatrix}\right)$ for $t \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$. Show that these form a group under matrix multiplication. More generally, show that if a set of matrices forms a group under multiplication, then either all matrices in the set have non-zero determinant, or all have zero determinant.

10. Let G be the set of all 3×3 real matrices of determinant 1 of the form

$$\begin{pmatrix} a & 0 & 0 \\ b & x & y \\ c & z & w \end{pmatrix}.$$

Verify that G is a group. Find a surjective homomorphism from G onto the group $GL_2(\mathbb{R})$ of all non-singular 2×2 real matrices, and find its kernel.

11. Exhibit a surjective homomorphism of the orthogonal group $\mathrm{O}(3)$ onto C_2 and another onto the special orthogonal group $\mathrm{SO}(3)$.

12. Find a subgroup of $\mathrm{GL}_2(\mathbb{R})$ which is isomorphic to D_8.
* Once you have done that, can you do a general D_{2n}?

The starred and exploration questions are not necessarily harder, but not necessary for a good understanding of the course. They should only be attempted once you have a solid understanding of the core material. They should also not be attempted to the detriment of later example sheets, or other courses. Exploration questions are meant to lead you as far as you are interested: just start and see how far you can get. There is not necessarily a "full solution".

13. * Consider a pack of $2n$ cards, numbered from 0 to $2n - 1$. An *outer perfect shuffle* is a shuffle of the cards, in which one first splits the pack in two halves of equal sizes and then interleaves the cards of the two halves in such a way that the top and bottom card remain in the top and bottom position. Show that the order of the outer shuffle is the multiplicative order of 2 modulo $2n - 1$.

Deduce that after at most $2n - 2$ repetitions of the outer shuffle we get the cards in the pack into the original position.

What is the actual order of the outer shuffle of the usual pack of 52 cards?

(There is also an *inner perfect shuffle* which differs from the outer shuffle in that the interleaving of the cards of the two halves is done so that neither the top nor the bottom card remains in the same position. What is the order of this shuffle of the usual pack of 52 cards?)

14. * Must a group of order 55 have elements of order 5 and order 11? Must a group of order 65 have elements of order 5 and order 13?

15. * *Exploration Question* Read more about frieze groups in the typeset lecture notes (which you can find on Moodle or on my personal website). Work through some of the arguments there.

In fact, the examples given there are the *only* possible frieze group examples. Can you prove this?

[Hints available on Moodle forum if you like. Reminder: Exploration question, so just go as far as you are interested.]