Category Theory Example Sheet 4

Michaelmas 2013 Julia Goedecke

These questions are of varying difficulty and length. Starred questions are not necessarily harder, but are extra questions. Comments, corrections and clarifications can be emailed to jg352. You can find this sheet on `www.dpmms.cam.ac.uk/~jg352/teaching.html`.

1. In a pointed category, show that $\ker(0\colon A \longrightarrow B) = 1_A$.

2. (a) Let $\mathcal{C}$ be a small category and $\mathcal{A}$ abelian. Show that the functor category $[\mathcal{C}, \mathcal{A}]$ is abelian.
   (b) Let $\mathcal{B}$ be small preadditive and $\mathcal{A}$ abelian. Prove that the full subcategory $\mathrm{Add}(\mathcal{B}, \mathcal{A}) \subset [\mathcal{B}, \mathcal{A}]$ of additive functors $\mathcal{B} \longrightarrow \mathcal{A}$ is abelian.
   (c) Show that for a (unitary) ring R, the category R-Mod of (left) R-modules is isomorphic to $\mathrm{Add}(R, \mathsf{AbGp})$.

3. **Additive Yoneda Lemma**
   (a) If $\mathcal{A}$ is a preadditive category and A is an object in $\mathcal{A}$, prove that the "representable functor" $\mathcal{A}(A, -)\colon \mathcal{A} \longrightarrow \mathsf{AbGp}$ is additive.
   (b) Given an object A in a preadditive $\mathcal{A}$ and $F\colon \mathcal{A} \longrightarrow \mathsf{AbGp}$, prove that there exists an isomorphism of abelian groups
   $$\theta_{A,F}\colon \mathrm{Nat}(\mathcal{A}(A, -), F) \cong F(A)$$
   which is natural in A and F.

4. A *pseudo-mono* is a morphism $f\colon A \longrightarrow B$ such that $fg = 0$ implies $g = 0$.
   (a) Show that if $\mathcal{A}$ is preadditive, then any pseudo-mono in $\mathcal{A}$ is a mono.
   (b) Let $\mathcal{C}$ be pointed with kernels and cokernels, such that every mono in $\mathcal{C}$ is normal. Show that every morphism in $\mathcal{C}$ factors as a pseudo-epi followed by a mono. [Given $f\colon A \longrightarrow B$, let $k = \ker \operatorname{coker}(f)$, and prove that the factorisation g of f over k is a pseudo-epi.]

5. (a) Show that in the category $\mathsf{AbGp}_{t.f.}$ of torsion-free abelian groups, not every monomorphism is a kernel and not every epimorphism is a cokernel. [Warning: epimorphisms in this category do not have to be surjective.]
   (b) Let $\mathcal{C}$ be the category of finitely-generated abelian groups having no elements of order 4 (though they may have elements of order 2), and homomorphisms between them. Show that every epimorphism in $\mathcal{C}$ is (surjective, and hence) a cokernel, but not every monomorphism in $\mathcal{C}$ is a kernel.
   (c) Let $\mathcal{A}$ be a preadditive category with kernels and cokernels, in which every epimorphism is a cokernel but not every monomorphism is a kernel. Show that normal monomorphisms (equivalently, regular monomorphisms) in $\mathcal{A}$ must fail to be closed under composition. [Given a non-normal monomorphism f, factor it as kg where k is the kernel of the cokernel of f; then let l be the kernel of the cokernel of g, and show that kl is not a normal monomorphism.]

6. Let $\mathcal{A}$ be a preadditive category and consider a reflexive pair $A \overset{f}{\underset{g}{\rightrightarrows}} B$ with common splitting $r\colon B \longrightarrow A$. Show that this reflexive pair (f, g) has the structure of an *internal groupoid*: this means that, for any object C in $\mathcal{A}$, the set $\mathcal{A}(C, B)$ is the set of objects of a groupoid, whose morphisms are the elements of $\mathcal{A}(C, A)$, with domain and codomain given by composition with f and g respectively. [Hint: Work out what the composable pairs of morphisms are, what the identity morphisms are, and how composition is defined. Then prove the axioms of a category, and then show that this category is in fact a groupoid.]

7. Let $\mathcal{A}$ be abelian. Consider

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array} \qquad \text{and} \qquad A \xrightarrow{\binom{f}{-g}} B \oplus C \xrightarrow{(h,k)} D$$

Prove

(a) $(h,k)\begin{pmatrix} f \\ -g \end{pmatrix} = 0$ iff the square commutes.

(b) $\begin{pmatrix} f \\ -g \end{pmatrix} = \ker(h,k)$ iff the square is a pullback.

(c) $(h,k) = \operatorname{coker}\begin{pmatrix} f \\ -g \end{pmatrix}$ iff the square is a pushout.

8. Let $\mathcal{A}$ be an abelian category. Prove that image factorisation is stable under pullback. This means: Given $f\colon A \longrightarrow B$ with image factorisation $A \xrightarrow{p} I \xrightarrow{i} B$ and two consecutive pullbacks

$$\begin{array}{ccccc} A' & \xrightarrow{p'} & I' & \xrightarrow{i'} & B' \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle b} \\ A & \xrightarrow[p]{} & I & \xrightarrow[i]{} & B \end{array}$$

then $i'p'$ is the image factorisation of the pullback of f along b.

9. In an abelian category, given a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{f} & A & \xrightarrow{g} & B & \longrightarrow & 0 \\ & & {\scriptstyle k}\downarrow & & {\scriptstyle a}\downarrow & (2) & \downarrow{\scriptstyle b} & & \\ 0 & \longrightarrow & K' & \xrightarrow[f']{} & A' & \xrightarrow[g']{} & B' & & \end{array}$$

where both rows are exact, use the Short Five Lemma to prove that if k is an iso then (2) is a pullback. [Hint: Try to do this first without looking in the latexed notes, but if you need some help, have a little peak and try to finish it yourself.]

10. (Pullback cancellation on the left) In an abelian category, consider

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle a}\downarrow & \lrcorner\ (1) & \downarrow{\scriptstyle b} & (2) & \downarrow{\scriptstyle c} \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' \end{array}$$

where the rectangle (1,2) and the square (1) are pullbacks and b is an epi. Use the previous question to prove that then (2) is also a pullback. [Hint: Try to do this first without looking in the latexed notes, but if you need some help, have a little peak and try to finish it yourself.]

11. ** *(Just in case you're interested and have read up the Nine Lemma.)* Use the Nine Lemma to prove the Noether's Third Isomorphism Theorem: In an abelian category $\mathcal{A}$, consider subobjects $A \rightarrowtail B \rightarrowtail C$. Then B/A is a subobject of C/A and $(C/A)/(B/A) \cong C/B$.