Category Theory Example Sheet 4

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These questions are of varying difficulty and length. Comments, corrections and clarifications can be emailed to jg352. You can find this sheet on `www.dpmms.cam.ac.uk/~jg352/teaching.html`.

1. In a pointed category, show that $\ker(0\colon A \longrightarrow B) = 1_A$.

2. (a) Let $\mathcal{C}$ be a small category and $\mathcal{A}$ abelian. Show that the functor category $[\mathcal{C}, \mathcal{A}]$ is abelian.
   (b) Let $\mathcal{B}$ be preadditive and $\mathcal{A}$ abelian. Prove that the full subcategory $\mathrm{Add}(\mathcal{B}, \mathcal{A}) \subset [\mathcal{B}, \mathcal{A}]$ of additive functors $\mathcal{B} \longrightarrow \mathcal{A}$ is abelian.
   (c) Show that for a (unitary) ring R, the category R-Mod of (left) R-modules is isomorphic to $\mathrm{Add}(R, \mathsf{AbGp})$.

3. **Additive Yoneda Lemma**
   (a) If $\mathcal{A}$ is a preadditive category and A is an object in $\mathcal{A}$, prove that the "representable functor" $\mathcal{A}(A, -)\colon \mathcal{A} \longrightarrow \mathsf{AbGp}$ is additive.
   (b) Given an object A in a preadditive $\mathcal{A}$ and $F\colon \mathcal{A} \longrightarrow \mathsf{AbGp}$, prove that there exists an isomorphism of abelian groups
   $$\theta_{A,F}\colon \mathrm{Nat}(\mathcal{A}(A,-), F) \cong F(A)$$
   which is natural in A and F.

4. A category $\mathcal{A}$ is called *semi-additive* if it is enriched in the category of monoids. In this question you will prove that in certain cases a semi-additive structure exists (and then it is unique, see proof for additive structures in lectures).
   (a) Let X be a set equipped with a distinguished element 0, and two binary operations $+$ and $*$ both of which have 0 as a (two-sided) identity element, and which satisfy the 'middle interchange law'
   $$(x + y) * (z + w) = (x * z) + (y * w)\ .$$
   Show that $+$ and $*$ coincide and that they are (it is?) associative and commutative (i.e., X is a commutative monoid). [This is a well-known piece of pure algebra, which I've included here in case you haven't seen it before.]
   (b) Now let $\mathcal{A}$ be a locally small pointed category with finite products and coproducts, where the product of any two objects coincides with their coproduct (more precisely, the functors $\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ sending (A, B) to $A \times B$ and to $A + B$ are naturally isomorphic). By considering the distinguished element $0\colon A \longrightarrow 0 \longrightarrow B$ of $\mathcal{A}(A, B)$ and the two binary operations on this set sending (f, g) to the composites
   $$A \xrightarrow{(1,1)} A \times A \cong A + A \xrightarrow{(f,g)} B \qquad \text{and} \qquad A \xrightarrow{(f,g)} B \times B \cong B + B \xrightarrow{(1,1)} B$$
   respectively, show that $\mathcal{A}$ is semi-additive.

5. A *pseudo-mono* is a morphism $f\colon A \longrightarrow B$ such that $fg = 0$ implies $g = 0$.
   (a) Show that if $\mathcal{A}$ is preadditive, then any pseudo-mono in $\mathcal{A}$ is a mono.
   (b) Let $\mathcal{C}$ be pointed with kernels and cokernels, such that every mono in $\mathcal{C}$ is normal. Show that every morphism in $\mathcal{C}$ factors as a pseudo-epi followed by a mono. [Given $f\colon A \longrightarrow B$, let $k = \ker\operatorname{coker}(f)$, and prove that the factorisation g of f over k is a pseudo-epi.]

6. The following 'addition-free' definition of an abelian category is often found in textbooks: $\mathcal{A}$ is abelian if it has a zero object, binary products, binary coproducts, kernels and cokernels, every monomorphism in $\mathcal{A}$ is a kernel and every epimorphism is a cokernel. Show that this definition is equivalent to the one given in lectures, along the following lines:

   (a) Show that $\mathcal{A}$ has pullbacks of pairs $(f\colon A \longrightarrow C, g\colon B \longrightarrow C)$ one of which is monic [hint: consider the kernel of qf, where q is the cokernel of g], and deduce that $\mathcal{A}$ has equalizers (and hence all finite limits).

   (b) Dually, $\mathcal{A}$ has all finite colimits. Now show that any pseudo-mono in $\mathcal{A}$ is monic. [If f is pseudo-monic and $fx = fy$, let q be a coequalizer of x and y: note that q is epic, and hence a cokernel of some morphism z, but f factors through q and hence z factors through $0 \longrightarrow A$.]

   (c) Given two objects A and B, consider the morphism $f\colon A + B \longrightarrow A \times B$ with matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Show that f is both monic and epic, and hence an isomorphism. [Hint: first show that $\iota_2\colon B \longrightarrow A + B$ is the kernel of $(1, 0)\colon A + B \longrightarrow A$.] Deduce that $\mathcal{A}$ has a semi-additive structure.

   (d) Finally, obtain the additive inverse of a morphism $f\colon A \longrightarrow B$ by considering the (multiplicative!) inverse of the morphism $A \oplus B \longrightarrow A \oplus B$ with matrix $\left(\begin{smallmatrix} 1 & 0 \\ f & 1 \end{smallmatrix}\right)$.

7. (a) Show that in the category $\mathsf{AbGp}_{t.f.}$ of torsion-free abelian groups, not every monomorphism is a kernel and not every epimorphism is a cokernel. [Warning: epimorphisms in this category do not have to be surjective.]

   (b) Let $\mathcal{C}$ be the category of finitely-generated abelian groups having no elements of order 4 (though they may have elements of order 2), and homomorphisms between them. Show that every epimorphism in $\mathcal{C}$ is (surjective, and hence) a cokernel, but not every monomorphism in $\mathcal{C}$ is a kernel.

   (c) Let$\mathcal{A}$ be a preadditive category with kernels and cokernels, in which every epimorphism is a cokernel but not every monomorphism is a kernel. Show that normal monomorphisms (equivalently, regular monomorphisms) in $\mathcal{A}$ must fail to be closed under composition. [Given a non-normal monomorphism f, factor it as kg where k is the kernel of the cokernel of f; then let l be the kernel of the cokernel of g, and show that kl is not a normal monomorphism.]

8. Let $\mathcal{A}$ be abelian. Consider

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array} \qquad \text{and} \qquad A \xrightarrow{\binom{f}{-g}} B \oplus C \xrightarrow{(h,k)} D$$

   Prove

   (a) $(h, k)\binom{f}{-g} = 0$ iff the square commutes.

   (b) $\binom{f}{-g} = \ker(h, k)$ iff the square is a pullback.

   (c) $(h, k) = \operatorname{coker}\binom{f}{-g}$ iff the square is a pushout.

9. Use the Nine Lemma to prove the Noether's Third Isomorphism Theorem: In an abelian category $\mathcal{A}$, consider subobjects $A \rightarrowtail B \rightarrowtail C$. Then B/A is a subobject of C/A and $(C/A)/(B/A) \cong C/B$.

10. Given a complex

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

   show that $H_n(C_\bullet) = 0$ iff $C_\bullet$ is exact at C_n.