Category Theory Example Sheet 3

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These questions are of varying difficulty and length. Starred questions are not necessarily harder, but are extra questions. Comments, corrections and clarifications can be emailed to jg352. You can find this sheet on `www.dpmms.cam.ac.uk/~jg352/teaching.html`.

1. Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be an adjunction $F \dashv G$ with unit η and counit ε. Show that the following conditions are equivalent.

   (i) $F\eta_A$ is an isomorphism for all objects A of $\mathcal{C}$.

   (ii) $\varepsilon_F A$ is an isomorphism for all A.

   (iii) $G\varepsilon_F A$ is an isomorphism for all A.

   (iv) $GF\eta_A = \eta_{GFA}$ for all A.

   (v) $GF\eta_{GB} = \eta_{GFGB}$ for all objects B of $\mathcal{D}$.

   (vi)–(x) The duals of (i)–(v).

   [Hint: if you take the conditions in the cyclic order indicated, all implications are trivial except for (v)⇒(vi) and its dual (x)⇒(i).] An adjunction with these properties is said to be *idempotent*.

2. A *complete semilattice* is a partially ordered set A in which every subset has a least upper bound (i.e. A is cocomplete when regarded as a category); a complete semilattice homomorphism is a mapping preserving (order and) arbitrary least upper bounds. Use the Adjoint Functor Theorem to show that

   (a) a poset A is a complete semilattice iff A^{op} is;

   (b) the category CSLat of complete semilattices and their homomorphisms is isomorphic to its opposite.

3. Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be an adjunction $(F \dashv G)$ with F faithful, and let $K \in \operatorname{ob}\mathcal{D}$ be a coseparator of $\mathcal{D}$. Show that GK is a coseparator of $\mathcal{C}$.

4. Recall from Sheet 2 the theories of widgets and chads.

   (a) Use the General Adjoint Functor Theorem to show that $U\colon \mathsf{Widget} \longrightarrow \mathsf{Set}$ has a left adjoint.

   (b) Do the same for the forgetful functor $\mathsf{Widget} \longrightarrow \mathsf{Chad}$.

5. (a) Let $\mathcal{C}$ be an arbitrary category. Show that the monoid of natural transformations from the identity functor $1_{\mathcal{C}}$ to itself is commutative. [This monoid is sometimes called the *centre* of the category $\mathcal{C}$; if you think about what it is when $\mathcal{C}$ is a group, you will see why.]

   (b) Deduce that if $1_{\mathcal{C}}$ has a monad structure $(1_{\mathcal{C}}, \eta, \mu)$, then η is an isomorphism.

   (c) Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor having a right adjoint G, such that there is some natural isomorphism (not necessarily the unit of the adjunction) between $1_{\mathcal{C}}$ and GF. Show that the unit is also an isomorphism, and deduce that F is full and faithful.

   (d) Let Idem be the category of sets equipped with an idempotent endomorphism (cf. Example (e) in Section 3A in your notes). Show that the forgetful functor $U\colon \mathsf{Idem} \longrightarrow \mathsf{Set}$ has a left adjoint F, and that there are functors G and H with GF and UH both isomorphic to the identity on Set, but that the unit of $(F \dashv U)$ is not an isomorphism.

6. We say a monad $\mathbb{T} = (T, \eta, \mu)$ is *idempotent* if μ is an isomorphism (cf. Question 5).

   (a) Suppose that $\mathcal{D}$ is a reflective subcategory of $\mathcal{C}$ (i.e. the inclusion has a left adjoint). Show that the monad $\mathbb{T}_\mathcal{D}$ on $\mathcal{C}$ induced be this adjunction is idempotent.

   (b) Show that if $\mathbb{T}$ is idempotent, then the full subcategory $\mathrm{Fix}(\mathbb{T}) \subseteq \mathcal{C}$, whose objects are those $A \in \mathcal{C}$ such that $\eta_A \colon A \longrightarrow TA$ is an isomorphism, is reflective in $\mathcal{C}$.

   (c) A subcategory $\mathcal{D} \subset \mathcal{C}$ is said to be *replete* if any object which is isomorphic to one in $\mathcal{D}$ is again in $\mathcal{D}$. Show that the assignations

   $$\mathbb{T} \longmapsto \mathrm{Fix}(\mathbb{T}) \qquad \text{and} \qquad (\mathcal{D} \subseteq \mathcal{C}) \longmapsto \mathbb{T}_\mathcal{D}$$

   induce a bijection between idempotent monads on $\mathcal{C}$ and reflective, replete subcategories of $\mathcal{C}$.

   (d) If $\mathbb{T}$ is an idempotent monad on $\mathcal{C}$, show that a $\mathbb{T}$-algebra structure on an object A is necessarily a two-sided inverse for η_A, and deduce that $\mathcal{C}^\mathbb{T}$ is isomorphic to $\mathrm{Fix}(\mathbb{T}) \subseteq \mathcal{C}$.

   (e) Show also that the Kleisli category $\mathcal{C}_\mathbb{T}$ is equivalent to $\mathrm{Fix}(\mathbb{T})$.

7.* Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$, and let $\mathcal{D}$ be an arbitrary category. Show that each of the functors $(F \longmapsto T \circ F) \colon [\mathcal{D}, \mathcal{C}] \longrightarrow [\mathcal{D}, \mathcal{C}]$ and $(G \longmapsto G \circ T) \colon [\mathcal{C}, \mathcal{D}] \longrightarrow [\mathcal{C}, \mathcal{D}]$ carries a monad structure, and that the categories of algebras for these two monads are respectively equivalent to $[\mathcal{D}, \mathcal{C}^\mathbb{T}]$ and to $[\mathcal{C}_\mathbb{T}, \mathcal{D}]$. [Hint for the second one: show that algebra structures on a functor G correspond to factorizations of G through $F_\mathbb{T} \colon \mathcal{C} \longrightarrow \mathcal{C}_\mathbb{T}$.]

7. Recall from Sheet 2 the theories of widgets and chads. Use the Monadicity Theorem to show that Widget is monadic over Set.

8. Let $\mathcal{C}$ be a well-powered category and $\mathbb{T} = (T, \eta, \mu)$ a monad on $\mathcal{C}$. Prove that the category $\mathcal{C}^\mathbb{T}$ of $\mathbb{T}$-algebras is well-powered.

10.* Prove that the Kleisli category $\mathcal{C}_\mathbb{T}$ is equivalent to the full subcategory of $\mathcal{C}^\mathbb{T}$ given by the free $\mathbb{T}$-algebras: those objects (A, α) which are isomorphic to (TB, μ_B) for some $B \in \mathrm{ob}\,\mathcal{C}$.