Category Theory Example Sheet 3

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These questions are of varying difficulty and length. Comments, corrections and clarifications can be emailed to jg352. You can find this sheet on `www.dpmms.cam.ac.uk/~jg352/teaching.html`.

1. Let $\mathcal{C}$ be a category with initial and terminal objects which are not isomorphic (e.g. $\mathcal{C} = \mathsf{Set}$), and let $\mathbf{n}$ denote an n-element totally ordered set (so that functors $\mathbf{n} \longrightarrow \mathcal{C}$ "are" composable strings of $n-1$ morphisms of $\mathcal{C}$). Show that there are functors $F_0, F_1, \ldots, F_{n+1} \colon [\mathbf{n}, \mathcal{C}] \longrightarrow [\mathbf{n+1}, \mathcal{C}]$ and $G_0, G_1, \ldots, G_n \colon [\mathbf{n+1}, \mathcal{C}] \longrightarrow [\mathbf{n}, \mathcal{C}]$ which form an adjoint string of length $2n+3$: that is,
$$(F_0 \dashv G_0 \dashv F_1 \dashv G_1 \dashv \cdots \dashv G_n \dashv F_{n+1}) .$$
Show also that this string is maximal, i.e. that F_0 has no left adjoint and F_{n+1} has no right adjoint. [Hint: recall that a functor with a left adjoint preserves any limits that exist.] Can you find a maximal string of adjoint functors of arbitrary even length?

2. Let $\mathcal{C}$ be a locally small category with coproducts. Prove that a functor $G \colon \mathcal{C} \longrightarrow \mathsf{Set}$ has a left adjoint if and only if it is representable.

3. Prove that the "discrete diagram" functor $\Delta \colon \mathcal{C} \longrightarrow [\mathcal{J}, \mathcal{C}]$ has a right (resp. left) adjoint if and only if $\mathcal{C}$ has limits (resp. colimits) of shape $\mathcal{J}$.

4. Let $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ and $G \colon \mathcal{D} \longrightarrow \mathcal{C}$ be functors, and suppose we are given natural transformations $\alpha \colon 1_{\mathcal{C}} \longrightarrow GF$, $\beta \colon FG \longrightarrow 1_{\mathcal{D}}$ such that the composite $G\beta \circ \alpha_G \colon G \longrightarrow GFG \longrightarrow G$ is the identity. Show that the composite $\beta_F \circ F\alpha \colon F \longrightarrow F$ is idempotent, and deduce that if $\mathcal{D}$ is Cauchy-complete (cf. Sheet 1, Question 4) then G has a left adjoint. By taking $\mathcal{C}$ to be the discrete category with one object and choosing $\mathcal{D}$ suitably, show that the conclusion may fail if $\mathcal{D}$ is not Cauchy-complete.

5. Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be an adjunction $F \dashv G$ with unit η and counit ε. Show that the following conditions are equivalent.
   (i) $F\eta_A$ is an isomorphism for all objects A of $\mathcal{C}$.
   (ii) $\varepsilon_F A$ is an isomorphism for all A.
   (iii) $G\varepsilon_F A$ is an isomorphism for all A.
   (iv) $GF\eta_A = \eta_{GFA}$ for all A.
   (v) $GF\eta_{GB} = \eta_{GFGB}$ for all objects B of $\mathcal{D}$.
   (vi)–(x) The duals of (i)–(v).

   [Hint: if you take the conditions in the cyclic order indicated, all implications are trivial except for (v)⇒(vi) and its dual (x)⇒(i).] An adjunction with these properties is said to be *idempotent*.

6. A *complete semilattice* is a partially ordered set A in which every subset has a least upper bound (i.e. A is cocomplete when regarded as a category); a complete semilattice homomorphism is a mapping preserving (order and) arbitrary least upper bounds. Use the Adjoint Functor Theorem to show that
   (a) a poset A is a complete semilattice iff A^{op} is;
   (b) the category CSLat of complete semilattices and their homomorphisms is isomorphic to its opposite.

7. Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be an adjunction $(F \dashv G)$ with F faithful, and let $K \in \operatorname{ob} \mathcal{D}$ be a coseparator of $\mathcal{D}$. Show that GK is a coseparator of $\mathcal{C}$.

8. Recall from Sheet 2 the theories of widgets and chads.

   (a) Use the General Adjoint Functor Theorem to show that $U\colon \mathsf{Widget} \longrightarrow \mathsf{Set}$ has a left adjoint.

   (b) Do the same for the forgetful functor $\mathsf{Widget} \longrightarrow \mathsf{Chad}$.

9. (a) Let $\mathcal{C}$ be an arbitrary category. Show that the monoid of natural transformations from the identity functor $1_{\mathcal{C}}$ to itself is commutative. [This monoid is sometimes called the *centre* of the category $\mathcal{C}$; if you think about what it is when $\mathcal{C}$ is a group, you will see why.]

   (b) Deduce that if $1_{\mathcal{C}}$ has a monad structure $(1_{\mathcal{C}}, \eta, \mu)$, then η is an isomorphism.

   (c) Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor having a right adjoint G, such that there is some natural isomorphism (not necessarily the unit of the adjunction) between $1_{\mathcal{C}}$ and GF. Show that the unit is also an isomorphism, and deduce that F is full and faithful.

   (d) Let Idem be the category of sets equipped with an idempotent endomorphism (cf. Example (e) in Section 3A in your notes). Show that the forgetful functor $U\colon \mathsf{Idem} \longrightarrow \mathsf{Set}$ has a left adjoint F, and that there are functors G and H with GF and UH both isomorphic to the identity on Set, but that the unit of $(F \dashv U)$ is not an isomorphism.

10. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$, and suppose T has a right adjoint R. Show that R has the structure of a comonad $\mathbb{R}$, such that the category of $\mathbb{R}$-coalgebras is isomorphic to the category of $\mathbb{T}$-algebras. Deduce that a functor with adjoints on both sides is monadic iff it is comonadic. [Hint: you can do this "directly" by showing millions of diagrams commute, but there is also a shorter conceptual way.]

11. We say a monad $\mathbb{T} = (T, \eta, \mu)$ is *idempotent* if μ is an isomorphism (cf. Question 5).

    (a) Suppose that $\mathcal{D}$ is a reflective subcategory of $\mathcal{C}$ (i.e. the inclusion has a left adjoint). Show that the monad $\mathbb{T}_{\mathcal{D}}$ on $\mathcal{C}$ induced be this adjunction is idempotent.

    (b) Show that if $\mathbb{T}$ is idempotent, then the full subcategory $\mathrm{Fix}(\mathbb{T}) \subseteq \mathcal{C}$, whose objects are those $A \in \mathcal{C}$ such that $\eta_A\colon A \longrightarrow TA$ is an isomorphism, is reflective in $\mathcal{C}$.

    (c) A subcategory $\mathcal{D} \subset \mathcal{C}$ is said to be *replete* if any object which is isomorphic to one in $\mathcal{D}$ is again in $\mathcal{D}$. Show that the assignations

    $$\mathbb{T} \longmapsto \mathrm{Fix}(\mathbb{T}) \qquad \text{and} \qquad (\mathcal{D} \subseteq \mathcal{C}) \longmapsto \mathbb{T}_{\mathcal{D}}$$

    induce a bijection between idempotent monads on $\mathcal{C}$ and reflective, replete subcategories of $\mathcal{C}$.

    (d) If $\mathbb{T}$ is an idempotent monad on $\mathcal{C}$, show that a $\mathbb{T}$-algebra structure on an object A is necessarily a two-sided inverse for η_A, and deduce that $\mathcal{C}^{\mathbb{T}}$ is isomorphic to $\mathrm{Fix}(\mathbb{T}) \subseteq \mathcal{C}$.

    (e) Show also that the Kleisli category $\mathcal{C}_{\mathbb{T}}$ is equivalent to $\mathrm{Fix}(\mathbb{T})$.

12. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$, and let $\mathcal{D}$ be an arbitrary category. Show that each of the functors $(F \longmapsto T \circ F)\colon [\mathcal{D}, \mathcal{C}] \longrightarrow [\mathcal{D}, \mathcal{C}]$ and $(G \longmapsto G \circ T)\colon [\mathcal{C}, \mathcal{D}] \longrightarrow [\mathcal{C}, \mathcal{D}]$ carries a monad structure, and that the categories of algebras for these two monads are respectively equivalent to $[\mathcal{D}, \mathcal{C}^{\mathbb{T}}]$ and to $[\mathcal{C}_{\mathbb{T}}, \mathcal{D}]$. [Hint for the second one: show that algebra structures on a functor G correspond to factorizations of G through $F_{\mathbb{T}}\colon \mathcal{C} \longrightarrow \mathcal{C}_{\mathbb{T}}$.]

13. Recall from Sheet 2 the theories of widgets and chads. Use the Monadicity Theorem to show that Widget is monadic over Set.

14. Let $\mathcal{C}$ be a well-powered category and $\mathbb{T} = (T, \eta, \mu)$ a monad on $\mathcal{C}$. Prove that the category $\mathcal{C}^{\mathbb{T}}$ of $\mathbb{T}$-algebras is well-powered.

15. Prove that the Kleisli category $\mathcal{C}_{\mathbb{T}}$ is equivalent to the full subcategory of $\mathcal{C}^{\mathbb{T}}$ given by the free $\mathbb{T}$-algebras: those objects (A, α) which are isomorphic to (TB, μ_B) for some $B \in \operatorname{ob}\mathcal{C}$.