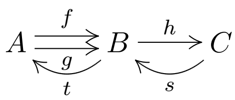
Category Theory Example Sheet 2

Michaelmas 2012 Julia Goedecke

These questions are of varying difficulty and length. Starred questions are not necessarily harder, but are extra questions. Comments, corrections and clarifications can be emailed to jg352. You can find this sheet on `www.dpmms.cam.ac.uk/~jg352/teaching.html`.

1. Let $(f\colon A \longrightarrow B, g\colon B \longrightarrow C)$ be a composable pair of morphisms.
   (a) Show that if both f and g are monic (resp. strong monic, split monic), then so is gf.
   (b) Show that if gf is monic (resp. strong monic, split monic), then so is f.
   (c) Show that if gf is regular monic and g is monic, then f is regular monic.
   (d) Formulate statements for epimorphisms corresponding to (b) and (c). Why don't you have to prove them?
   (e) Show that, in the category of commutative rings, the inclusion map $\mathbb{Z} \longrightarrow \mathbb{Q}$ is an epimorphism.
   (f) Deduce that functors need not preserve epimorphisms.
   (g) Deduce that functors need not preserve monomorphisms.
   (h) Show that the representable functor $\mathcal{C}(A, -)\colon \mathcal{C} \longrightarrow \mathsf{Set}$ preserves monomorphisms.

2. Consider the diagram



(∗)

   satisfying $hf = hg$, $hs = 1_C$, $gt = 1_B$ and $ft = sh$.
   (a) Show that h is a coequaliser for the pair (f, g). [(∗) is called a *split coequaliser diagram*.]
   (b) Which kind of functors preserve split coequalisers?
   (c) Recall from Sheet 1 that an idempotent e splits if it can be factored as fg where gf is an identity morphism. Show that an idempotent e splits iff the pair $(e, 1_{\mathrm{dom}\, e})$ has an equaliser, iff the same pair has a coequaliser.

3. In this question, you can either use the first two parts to give you ideas for Part (d), or you can just do Part (d) directly and leave out the others. Here $\mathcal{C}$ is a locally small category, and we are looking at representables $\mathcal{C}(A, -)\colon \mathcal{C} \longrightarrow \mathsf{Set}$.
   (a) Show that representables preserve (small) products.
   (b) Show that representables preserve equalisers.
   (c) Deduce that if $\mathcal{C}$ has equalisers and small products, then representables preserve small limits.
   (d) * Show that representables preserve all limits.

4. Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor.
   (a) Show that if $\mathcal{D}$ has and F creates limits of shape $\mathcal{J}$, then $\mathcal{C}$ has and F preserves them.
   (b) Show that if F creates limits of shape $\mathcal{J}$, then F reflects them.

5.* (a) A category $\mathcal{J}$ is said to be *connected* if it has just one connected component, i.e. if any two objects of $\mathcal{J}$ may be linked by a 'zig-zag' of morphisms. Show that a category has all finite connected limits (that is, limits of diagrams whose shapes are finite connected categories) iff it has pullbacks and equalizers.
   (b) Let $\mathcal{D}$ be a category, and $B \in \mathrm{ob}\,\mathcal{D}$. Show that the forgetful functor $\mathcal{D}/B \longrightarrow \mathcal{D}$ creates connected limits.

6. (a) A *widget* is a set A equipped with elements $0, 1 \in A$, a ternary operation $[-,-,-]\colon A^3 \longrightarrow A$, and for each rational number λ a unary operation $\lambda \cdot - \colon A \longrightarrow A$ satisfying the axioms

$$\begin{aligned} [a,0,b] &= [\frac{3}{4} \cdot b, \frac{1}{4} \cdot a, 1] \\ [[a,b,a],c,a] &= 0 \\ \lambda \cdot (\mu \cdot a) &= (\lambda\mu) \cdot a \\ (\lambda + \mu) \cdot a &= [\lambda \cdot a, \mu \cdot a, (5-\mu) \cdot a] \end{aligned}$$

for all $a, b, c \in A$ and rational λ, μ. Let Widget be the category of widgets and their homomorphisms. Show that the forgetful functor $U\colon \mathsf{Widget} \longrightarrow \mathsf{Set}$ creates limits. Deduce that Widget has all small limits and U preserves them.

(b) A *chad* is a set A equipped with an element $0 \in A$ and a ternary operation $[-,-,-]\colon A^3 \longrightarrow A$ satisfying the axiom

$$[[a,a,a],a,a] = 0$$

for all $a \in A$. Show that the forgetful functor $\mathsf{Widget} \longrightarrow \mathsf{Chad}$ creates small limits.

[*The theory of widgets is a typical finitary algebraic theory, and your proof of (a) should apply equally well to categories of algebras such as rings, groups, Lie algebras, modules, etc... Similarly, your proof of (b) should apply to all functors between such categories which arise from forgetting part of a theory.*]

7. Let $\mathcal{C}$ be a category with initial and terminal objects which are not isomorphic (e.g. $\mathcal{C} = \mathsf{Set}$), and let $\mathbf{n}$ denote an n-element totally ordered set (so that functors $\mathbf{n} \longrightarrow \mathcal{C}$ "are" composable strings of $n-1$ morphisms of $\mathcal{C}$). Show that there are functors $F_0, F_1, \ldots, F_{n+1}\colon [\mathbf{n}, \mathcal{C}] \longrightarrow [\mathbf{n+1}, \mathcal{C}]$ and $G_0, G_1, \ldots, G_n\colon [\mathbf{n+1}, \mathcal{C}] \longrightarrow [\mathbf{n}, \mathcal{C}]$ which form an adjoint string of length $2n+3$: that is,

$$(F_0 \dashv G_0 \dashv F_1 \dashv G_1 \dashv \cdots \dashv G_n \dashv F_{n+1})\ .$$

Show also that this string is maximal, i.e. that F_0 has no left adjoint and F_{n+1} has no right adjoint. [Hint: recall that a functor with a left adjoint preserves any limits that exist.] Can you find a maximal string of adjoint functors of arbitrary even length?

8. Let $\mathcal{C}$ be a locally small category with coproducts. Prove that a functor $G\colon \mathcal{C} \longrightarrow \mathsf{Set}$ has a left adjoint if and only if it is representable.

9. Prove that the "discrete diagram" functor $\Delta\colon \mathcal{C} \longrightarrow [\mathcal{J}, \mathcal{C}]$ has a right (resp. left) adjoint if and only if $\mathcal{C}$ has limits (resp. colimits) of shape $\mathcal{J}$.

10.* Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ and $G\colon \mathcal{D} \longrightarrow \mathcal{C}$ be functors, and suppose we are given natural transformations $\alpha\colon 1_{\mathcal{C}} \longrightarrow GF$, $\beta\colon FG \longrightarrow 1_{\mathcal{D}}$ such that the composite $G\beta \circ \alpha_G\colon G \longrightarrow GFG \longrightarrow G$ is the identity. Show that the composite $\beta_F \circ F\alpha\colon F \longrightarrow F$ is idempotent, and deduce that if $\mathcal{D}$ is Cauchy-complete (cf. Sheet 1, Question 4) then G has a left adjoint. By taking $\mathcal{C}$ to be the discrete category with one object and choosing $\mathcal{D}$ suitably, show that the conclusion may fail if $\mathcal{D}$ is not Cauchy-complete.