Category Theory Example Sheet 2

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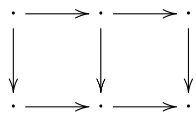
These questions are of varying difficulty and length. Comments, corrections and clarifications can be emailed to jg352. You can find this sheet on `www.dpmms.cam.ac.uk/~jg352/teaching.html`.

1. Prove that limits are unique up to unique isomorphism.

2. Consider a commutative diagram of the following form:

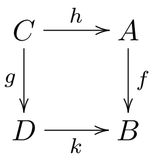


   (a) Show that if both small squares are pullbacks, then so is the large rectangle.

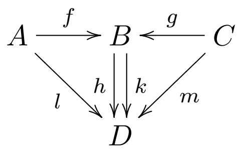
   (b) Show that if the large rectangle and the right hand square are pullbacks, then so is the left hand square.

   (c) Deduce from the above (or prove directly) that the pullback of a pullback square is a pullback, stating clearly what you take this to mean. [Consider a cube.]

3. A monomorphism $f\colon A \longrightarrow B$ in a category is said to be *strong* if, for every commutative square



   with g epic, there exists a (necessarily unique) $t\colon D \longrightarrow A$ such that $ft = k$ and $tg = h$. Show that every regular monomorphism is strong, but that in the finite category represented by the diagram



   the morphism f is strong monic but not regular monic.

4. Let $(f\colon A \longrightarrow B, g\colon B \longrightarrow C)$ be a composable pair of morphisms.

   (a) Show that if both f and g are monic (resp. strong monic, split monic), then so is gf.

   (b) Show that if gf is monic (resp. strong monic, split monic), then so is f.

   (c) Show that if gf is regular monic and g is monic, then f is regular monic.

   (d) Formulate statements for epimorphisms corresponding to (b) and (c). Why don't you have to prove them?

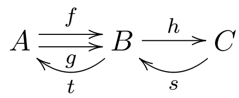
   (e) Show that, in the category of commutative rings, the inclusion map $\mathbb{Z} \longrightarrow \mathbb{Q}$ is an epimorphism.

   (f) Deduce that functors need not preserve epimorphisms.

   (g) Deduce that functors need not preserve monomorphisms.

   (h) Show that the representable functor $\mathcal{C}(A, -)\colon \mathcal{C} \longrightarrow \mathsf{Set}$ preserves monomorphisms.

5. Consider the diagram



(∗)

satisfying $hf = hg$, $hs = 1_C$, $gt = 1_B$ and $ft = sh$.

(a) Show that h is a coequaliser for the pair (f, g). [(∗) is called a *split coequaliser diagram.*]

(b) Which kind of functors preserve split coequalisers?

6. In this question, you can either use the first two parts to give you ideas for Part (d), or you can just do Part (d) directly and leave out the others. Here $\mathcal{C}$ is a locally small category, and we are looking at representables $\mathcal{C}(A, -)\colon \mathcal{C} \longrightarrow \mathsf{Set}$.

(a) Show that representables preserve (small) products.

(b) Show that representables preserve equalisers.

(c) Deduce that if $\mathcal{C}$ has equalisers and small products, then representables preserve small limits.

(d) Show that representables preserve all limits.

7. Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor.

(a) Show that if $\mathcal{D}$ has and F creates limits of shape $\mathcal{J}$, then $\mathcal{C}$ has and F preserves them.

(b) Show that if F creates limits of shape $\mathcal{J}$, then F reflects them.

8. (a) A category $\mathcal{J}$ is said to be *connected* if it has just one connected component, i.e. if any two objects of $\mathcal{J}$ may be linked by a ‘zig-zag’ of morphisms. Show that a category has all finite connected limits (that is, limits of diagrams whose shapes are finite connected categories) iff it has pullbacks and equalizers.

(b) Let $\mathcal{D}$ be a category, and $B \in \operatorname{ob}\mathcal{D}$. Show that the forgetful functor $\mathcal{D}/B \longrightarrow \mathcal{D}$ creates connected limits.

9. (a) A *widget* is a set A equipped with elements $0, 1 \in A$, a ternary operation $[-,-,-]\colon A^3 \longrightarrow A$, and for each rational number λ a unary operation $\lambda \cdot -\colon A \longrightarrow A$ satisfying the axioms

$$\begin{aligned} [a, 0, b] &= [\frac{3}{4} \cdot b, \frac{1}{4} \cdot a, 1] \\ [[a, b, a], c, a] &= 0 \\ \lambda \cdot (\mu \cdot a) &= (\lambda\mu) \cdot a \\ (\lambda + \mu) \cdot a &= [\lambda \cdot a, \mu \cdot a, (5 - \mu) \cdot a] \end{aligned}$$

for all $a, b, c \in A$ and rational λ, μ. Let Widget be the category of widgets and their homomorphisms. Show that the forgetful functor $U\colon \mathsf{Widget} \longrightarrow \mathsf{Set}$ creates limits. Deduce that Widget has all small limits and U preserves them.

(b) A *chad* is a set A equipped with an element $0 \in A$ and a ternary operation $[-,-,-]\colon A^3 \longrightarrow A$ satisfying the axiom

$$[[a, a, a], a, a] = 0$$

for all $a \in A$. Show that the forgetful functor $\mathsf{Widget} \longrightarrow \mathsf{Chad}$ creates limits.

[*The theory of widgets is a typical finitary algebraic theory, and your proof of (a) should apply equally well to categories of algebras such as rings, groups, Lie algebras, modules, etc... Similarly, your proof of (b) should apply to all functors between such categories which arise from forgetting part of a theory.*]