Category Theory Example Sheet 1

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These questions are of varying difficulty and length. Comments, corrections and clarifications can be emailed to jg352. You can find this sheet on `www.dpmms.cam.ac.uk/~jg352/teaching.html`.

1. (a) Show that identities in a category are unique.
   (b) Show that a morphism with both a right inverse and a left inverse is an isomorphism.
   (c) Consider $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$. Show that if two out of f, g and gf are isomorphisms, then so is the third. [This is known as the *two-out-of-three property*.]
   (d) Show that functors preserve isomorphisms.
   (e) Show that if $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is full and faithful, and $Ff\colon FA \longrightarrow FB$ is an isomorphism in $\mathcal{D}$, then $f\colon A \longrightarrow B$ is an isomorphism in $\mathcal{C}$. [In this case we say F *reflects isomorphisms*.]

2. (a) Show that there is a functor $\mathrm{ob}\colon \mathsf{Cat} \longrightarrow \mathsf{Set}$ sending a small category to its set of objects. Is it faithful? Is it full?
   (b) Show that there is a functor $\mathrm{mor}\colon \mathsf{Cat} \longrightarrow \mathsf{Set}$ sending a small category to its set of morphisms. Is it faithful? Is it full?
   (c) Show that the domain and codomain operations give rise to two natural transformations $\mathrm{dom}, \mathrm{cod}\colon \mathrm{mor} \longrightarrow \mathrm{ob}$.

3. Let $\mathcal{G}$ be a group viewed as a one-object category. Show that the nat. transformations $\alpha\colon 1_{\mathcal{G}} \longrightarrow 1_{\mathcal{G}}$ correspond to elements in the centre of the group.

4. A morphism $e\colon A \longrightarrow A$ is called *idempotent* if $ee = e$. An idempotent e is said to *split* if it can be factored as fg where gf is an identity morphism.
   (a) Let $\mathcal{E}$ be a class of idempotents in a category $\mathcal{C}$. Show that there is a category $\mathcal{C}[\check{\mathcal{E}}]$ whose objects are the members of $\mathcal{E}$, whose morphisms $e \longrightarrow d$ are those morphisms $f\colon \mathrm{dom}\, e \longrightarrow \mathrm{dom}\, d$ in $\mathcal{C}$ for which $dfe = f$, and whose composition coincides with composition in $\mathcal{C}$. [*Warning*: the identity morphism on an object e is not $1_{\mathrm{dom}\, e}$, in general.]
   (b) If $\mathcal{E}$ is a class of idempotents containing all identity morphisms of $\mathcal{C}$, show that there is a full and faithful functor $I\colon \mathcal{C} \longrightarrow \mathcal{C}[\check{\mathcal{E}}]$, and that an arbitrary functor $T\colon \mathcal{C} \longrightarrow \mathcal{D}$ can be factored as $\widehat{T}I$ for some $\widehat{T}$ iff it sends the members of $\mathcal{E}$ to split idempotents in $\mathcal{D}$.
   (c) If all idempotents in $\mathcal{C}$ split, $\mathcal{C}$ is said to be *Cauchy-complete*; the *Cauchy-completion* $\widehat{\mathcal{C}}$ of an arbitrary category $\mathcal{C}$ is defined to be $\mathcal{C}[\check{\mathcal{E}}]$, where $\mathcal{E}$ is the class of all idempotents in $\mathcal{C}$. Verify that the Cauchy-completion of a category is indeed Cauchy-complete.

5. (a) Show that any functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ can be factorised as

$$\mathcal{C} \xrightarrow{L} \mathcal{E} \xrightarrow{R} \mathcal{D}$$

   where L is bijective on objects, and R is full and faithful.
   (b) Show that, in a commuting square of functors

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle R} \\ \mathcal{C} & \xrightarrow[G]{} & \mathcal{E} \end{array}$$

   with L bijective on objects and R full and faithful, there exists a unique functor $J\colon \mathcal{C} \longrightarrow \mathcal{D}$ with $JL = F$ and $RJ = G$.

(c) Deduce that a functor which is both bijective on objects and full and faithful is an isomorphism of categories.

(d) Deduce that the factorisation in (a) is unique up to unique isomorphism, stating clearly what you take this to mean.

6. Show that the category Set_* of pointed sets is equivalent to the category Part of sets and partial functions.

7. Let L be a distributive lattice (i.e. a partially ordered set with finite joins (suprema, $\vee$) and meets (infima, $\wedge$), satisfying the distributive law

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

for all $a, b, c \in L$). Show that there is a category Mat_L whose objects are the natural numbers, and whose morphisms $n \longrightarrow m$ are $m \times n$ matrices with entries from L, where we define 'multiplication' of such matrices by analogy with that of matrices over a field, interpreting $\wedge$ as multiplication and $\vee$ as addition. Show also that if L is the two-element lattice $\{0, 1\}$ with $0 \leq 1$, then Mat_L is equivalent to the category Rel_f of finite sets and relations between them.

8. Prove that $\theta\colon \mathsf{Nat}(\mathcal{C}(A, -), F) \longrightarrow FA$ from the Yoneda Lemma is natural in F for fixed A.

9. Let $\mathcal{C}$ be a small category, and $F, G\colon \mathcal{C} \longrightarrow \mathsf{Set}$ two functors. Use the Yoneda Lemma to show that a natural transformation $\alpha\colon F \longrightarrow G$ is a monomorphism in $[\mathcal{C}, \mathsf{Set}]$ if and only if all components α_A are monomorphisms in Set.

10. By an *automorphism* of a category $\mathcal{C}$, we of course mean a functor $F\colon \mathcal{C} \longrightarrow \mathcal{C}$ with a (2-sided) inverse. We say an automorphism F is *inner* if it is naturally isomorphic to the identity functor. [To see the justification for this name, think about the case when $\mathcal{C}$ is a group.]

(a) Show that the inner automorphisms of $\mathcal{C}$ form a normal subgroup of the group of all automorphisms of $\mathcal{C}$. [Don't worry about whether these groups are sets or proper classes.]

(b) If F is an automorphism of a category $\mathcal{C}$ with a terminal object 1, show that $F(1)$ is also a terminal object of $\mathcal{C}$ (and hence isomorphic to 1).

(c) Deduce that, for any automorphism F of Set, there is a *unique* natural isomorphism from the identity to F. [*Hint*: Yoneda]

11. Find representations for the following functors. (All functors are defined on morphisms in the only sensible way.)

(a) For fixed sets A and B, the functor

$$\begin{aligned} \mathsf{Set}^{\mathrm{op}} &\longrightarrow \mathsf{Set} \\ X &\longmapsto \{\text{pairs of functions } f\colon X \longrightarrow A \text{ and } g\colon X \longrightarrow B\}. \end{aligned}$$

(b) For fixed morphisms $f, g\colon A \longrightarrow B$ in the category Gp, the functor

$$\begin{aligned} \mathsf{Gp}^{\mathrm{op}} &\longrightarrow \mathsf{Set} \\ G &\longmapsto \{\text{morphisms } h\colon G \longrightarrow A \text{ with } fh = gh\}. \end{aligned}$$

(c) For a commutative ring R and an ideal I in R, the functor

$$\begin{aligned} \mathsf{CRng} &\longrightarrow \mathsf{Set} \\ S &\longmapsto \{\text{homomorphisms } f\colon R \longrightarrow S \text{ with } f(I) = 0\}. \end{aligned}$$