Further Examples

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This handout gives examples of separating and coseparating sets. They are taken from different areas of maths. You can find this sheet on www.dpmms.cam.ac.uk/~jg352/teaching.html.

Coseparating sets

- For any category $\mathcal{C}$, $\mathrm{ob}\,\mathcal{C}$ is a coseparating family, so if $\mathcal{C}$ is small, it is a coseparating set.
- The category Set has coseparator (i.e. singleton coseparating set) $2 = \{0, 1\}$. Given $A \underset{g}{\overset{f}{\rightrightarrows}} B$ with $f \neq g$, we have an $a \in A$ such that $f(a) \neq g(a)$. Then for example the characteristic function of $\{f(a)\}$ satisfies $\chi_{\{f(a)\}} f \neq \chi_{\{f(a)\}} g$. In the proof of the Special Adjoint Functor Theorem, if you take $\mathcal{A} = \mathsf{Set}$, then what we were doing is finding the intersection of all subsets of 2 to see it is an initial object. But of course that is the empty set, which is indeed an initial object.
- For any field k, the category of vector-spaces k-Mod has coseparator k. The intersection of all subobjects of k is of course the zero space, which is an initial object.
- The category of affine spaces over a field k has the affine line k as a coseparator.
- The slice category k-Mod/k has the addition $k \oplus k \longrightarrow k$ as a coseparator. In fact, this generalises both previous examples: you can get back the example of k-Mod by taking the kernels of all the zero morphisms in this slice category and the coseparator is the kernel of the addition, and the full subcategory with objects the epimorphisms is equivalent to the category of affine spaces over k (and the addition $k \oplus k \longrightarrow k$ goes to the affine line k).
- The space $[0, 1]$ is a coseparator in KHaus, by Uryson's Lemma. This is part of the ingredients to use the Special Adjoint Functor Theorem to construct the Stone-Čech compactification functor.

Separating families

- For any $\mathcal{C}$, $\mathrm{ob}\,\mathcal{C}$ is a separating family.
- For any locally small $\mathcal{C}$, the collection $\{Y(A) |\ A \in \mathrm{ob}\,\mathcal{C}\}$ is a separating family in $[\mathcal{C}, \mathsf{Set}]$: If $F \underset{\beta}{\overset{\alpha}{\rightrightarrows}} G$ satisfies $\alpha \neq \beta$, there there exists $A \in \mathrm{ob}\,\mathcal{C}$ and $x \in FA$ with $\alpha_A(x) \neq \beta_A(x)$. Then $\psi(x) \colon Y(A) \longrightarrow F$ satisfies $\alpha\psi(x) \neq \beta\psi(x)$.
- In Set, $1 = \{*\}$ is a separating set, since $\mathsf{Set}(1, -)$ is isomorphic to the identity functor.
- In Gp, $\mathbb{Z}$ is a separator, since $\mathsf{Gp}(\mathbb{Z}, -)$ is isomorphic to the forgetful functor.
- In Top, 1 is a separator, since $\mathsf{Top}(1, -)$ is faithful.