Further Examples

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This handout gives additional examples of adjunctions. They are taken from different areas of maths, so if you know that area, try to prove that what I claim really is an adjunction. A lot of these examples come from MacLane's Categories for the Working Mathematician, so you can read up on them there as well (III.1 and IV.2). You can find this sheet on `www.dpmms.cam.ac.uk/~jg352/teaching.html`.

Adjunctions

- $U\colon \mathsf{AbGp} \longrightarrow \mathsf{Gp}$ has left adjoint the abelianisation functor $\mathrm{ab}\colon \mathsf{Gp} \longrightarrow \mathsf{AbGp}$.
- $U\colon \mathsf{Alg}_k \longrightarrow k\text{-}\mathsf{Mod}$ forgetting the multiplicative structure has a left adjoint

$$\begin{aligned} F\colon k\text{-}\mathsf{Mod} &\longrightarrow \mathsf{Alg}_k \\ V &\longmapsto \text{free } k\text{-algebra on } V \text{ (tensor algebra)} \end{aligned}$$

- $U\colon \mathsf{Alg}_k \longrightarrow \mathsf{Lie}_k$ has left adjoint "universal enveloping algebra".
- $U\colon R\text{-}\mathsf{Mod} \longrightarrow \mathsf{AbGp}$ has a left adjoint sending an abelian group A to $R\otimes A$. In fact, U is representable, i.e. $U \cong \mathrm{Hom}_R(R, -) = R\text{-}\mathsf{Mod}(R, -)$. The module homomorphisms between any two modules form an abelian group (under pointwise addition), so it lands in AbGp.

  In fact, this is a special case of a "hom-tensor adjunction":
- Given a module M, we have an adjunction $M\otimes_R - \dashv \mathrm{Hom}_R(M, -)$, where both functors go from $R\text{-}\mathsf{Mod}$ to $R\text{-}\mathsf{Mod}$. Of course, $\mathrm{Hom}_R(M, A)$ is an R-module by pointwise addition and scalar multiplication. The bijection of the adjunction is: $f\colon M\otimes_R A \longrightarrow B$ corresponds to the morphism $A \longrightarrow \mathrm{Hom}_R(M, B)$ sending $a \in A$ to $g_a\colon M \longrightarrow B$ with $g_a(m) = f(m\otimes a)$.
- Our original $U\colon R\text{-}\mathsf{Mod} \longrightarrow \mathsf{AbGp}$ also has a right adjoint sending an abelian group A to the R-module $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ of group homomorphisms (i.e. $\mathbb{Z}$-module homomorphisms) from UR to A. Here $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ is obviously an abelian group, but it can be given an R-module structure as follows: given $f\colon R \longrightarrow A$ and $r \in R$, define the function rf by $rf(x) = f(rx)$.

  More in the same spirit:
- Let $R\text{-}\mathsf{Mod}\text{-}S$ be the category of left R- and right S-modules. Then $U\colon R\text{-}\mathsf{Mod}\text{-}S \longrightarrow R\text{-}\mathsf{Mod}$ has a left adjoint sending an R-module M to $M\otimes S$.
- Given a ring homomorphism $\phi\colon S \longrightarrow R$, we can view any R-module M as an S-module via $sm = \phi(s)m$. Then the forgetful functor $R\text{-}\mathsf{Mod} \longrightarrow S\text{-}\mathsf{Mod}$ is $\mathrm{Hom}_R(R, -)$, where the S-module R gives $\mathrm{Hom}_R(R, A)$ an S-module structure. This has a left adjoint $R\otimes_S -$.
- The forgetful functor from small categories to graphs has a left adjoint sending a graph G to the free category on G (i.e. you have to "add" identities and compositions).
- The inclusion $\mathsf{KHaus} \longrightarrow \mathsf{Top}$ of compact Hausdorff spaces into Top has a left adjoint, the Stone-Čech compactification functor. In fact, Čech's original proof went very much along the lines of the proof of the Special Adjoint Functor Theorem.
- Let $\mathcal{H} \subset \mathcal{G}$ be groups regarded as one-object categories, and denote the inclusion functor by $I\colon \mathcal{H} \longrightarrow \mathcal{G}$. Recall that $[\mathcal{G}, k\text{-}\mathsf{Mod}]$ is the category of k-linear representations of $\mathcal{G}$. We have a functor "restriction"

$$\begin{aligned} [\mathcal{G}, k\text{-}\mathsf{Mod}] &\longrightarrow [\mathcal{H}, k\text{-}\mathsf{Mod}] \\ R &\longmapsto RI \end{aligned}$$

  and this has a left adjoint "induction".

- The forgetful functor from the category Fld of fields to the category Dom_m of integral domains and monomorphisms between integral domains has left adjoint "field of quotients". (Note that any homomorphism between fields is a mono.) This doesn't work if you take all morphisms between integral domains! (see MacLane III.1 for more explanation)

- Let X be a set. Then the functor
$$- \times X \colon \mathsf{Set} \longrightarrow \mathsf{Set}$$
has a right adjoint
$$(-)^X \colon \mathsf{Set} \longrightarrow \mathsf{Set}.$$
Here A^X is the set of functions from X to A. This is an example of an "internal hom": a hom viewed as an object of the category itself. A category with products where $- \times X$ has a right adjoint is called *cartesian closed*. The "hom-tensor" adjunction above is similar, there we say the category is *monoidal closed*.

- Let $\mathcal{C}$ be a small category. Then the functor
$$- \times \mathcal{C} \colon \mathsf{Cat} \longrightarrow \mathsf{Cat}$$
has a right adjoint
$$[\mathcal{C}, -] \colon \mathsf{Cat} \longrightarrow \mathsf{Cat}.$$

- Let X be a locally compact space. Then
$$- \times X \colon \mathsf{Top} \longrightarrow \mathsf{Top}$$
has a right adjoint
$$(-)^X \colon \mathsf{Top} \longrightarrow \mathsf{Top}$$
where Y^X has compact open topology.

- Let V be a vector space. Then $- \otimes V \dashv \mathcal{L}(V, -)$ (as functors from k-Mod to itself), where $\mathcal{L}(V, W)$ is the vector space of linear maps from V to W.

- Given a map $f \colon X \longrightarrow Y$ between topological spaces, the "pushforward" or "direct image" functor is left adjoint to the "pullback" or "inverse image" functor between the categories of sheaves. There are a lot of (more general) direct image and inverse image functors in Topos theory.

- Regard a monoid M as a discrete category, with elements of M as objects. Then the multiplication of M gives a functor $\mu \colon M \times M \longrightarrow M$. If M is a group, the group inverse provides right adjoints for the functors $\mu(x, -)$ and $\mu(-, y) \colon M \longrightarrow M$. Conversely, does the presence of such adjoints make M into a group?