Further Examples

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This handout gives additional examples of categories and functors. They are taken from different areas of maths, so if you don't know what something is, don't worry about it. You can find this sheet on `www.dpmms.cam.ac.uk/~jg352/teaching.html`.

Categories

- $\mathsf{Set}_{\text{inj}}$: objects are sets, morphisms are injective functions.
- Poset: objects are posets, morphisms are orderpreserving functions.
- CRng: commutative rings and ring homomorphisms.
- Alg_k of k-algebras, Lie_k of Lie algebras, etc...
- Δ, the simplicial category: objects are non-empty finite ordinals $[n] = \{0 \leq 1 \leq \cdots \leq n\}$ and morphisms are order-preserving maps.
- There is a category whose objects are finite cardinals $n = \{0, 1, \cdots, n-1\}$ and whose morphisms are bijections.
- Given a topological space X, the set $\mathcal{O}(X)$ of open sets of X with inclusion as a partial order is a category (as a poset).
- There are categories of affine varieties over a field k, of quasi-projective or abstract varieties over a field k, ...
- $\mathsf{Sh}(X)$: category of sheaves over a topological space X (of course you have to specify what kind of sheaves).
- A category whose objects are Banach spaces and whose morphisms are linear maps which are contractive, or distance-decreasing.
- Make up your own.

Functors

- Given a commutative ring R, there is a functor $\mathsf{Gp} \longrightarrow \mathsf{CRng}$ sending G to RG, the group ring.
- Constant functors: Given categories $\mathcal{C}$ and $\mathcal{D}$ and $D \in \operatorname{ob} \mathcal{D}$, there is a constant functor

$$\begin{aligned} \Delta_D \colon \mathcal{C} &\longrightarrow \mathcal{D} \\ A &\longmapsto D \\ f &\longmapsto 1_D \end{aligned}$$

  There is also a functor sending D to the constant functor on D:

$$\begin{aligned} \Delta \colon \mathcal{D} &\longrightarrow [\mathcal{C}, \mathcal{D}] \\ D &\longmapsto \Delta_D \\ f &\longmapsto \alpha \end{aligned}$$

  with $\alpha_A = f \colon D \longrightarrow E$.

- evaluation functors: Given two categories $\mathcal{C}$ and $\mathcal{D}$ and an object A in $\mathcal{C}$, we have the functor

$$\begin{aligned} \mathrm{ev}_A \colon [\mathcal{C}, \mathcal{D}] &\longrightarrow \mathcal{D} \\ F &\longmapsto FA \\ \alpha &\longmapsto \alpha_A \end{aligned}$$

  We can also put all these functors together:

$$\begin{aligned} \mathrm{ev} \colon \mathcal{C} \times [\mathcal{C}, \mathcal{D}] &\longrightarrow \mathcal{D} \\ (A, F) &\longmapsto FA \\ (f, \alpha) &\longmapsto \alpha_B Ff = Gf\alpha_A \end{aligned}$$

- A functor $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Set}$ is called a **presheaf**. Functors $\mathcal{C} \longrightarrow \mathsf{Set}$ are also sometimes called covariant presheaves. The hom-functor $\mathcal{C}(-, A) \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Set}$ is a presheaf.

- Open sets functor (this is a presheaf):

$$\begin{aligned} \mathcal{O}(-) \colon \mathsf{Top}^{\mathrm{op}} &\longrightarrow \mathsf{Set} \\ X &\longmapsto \{\text{open sets of } X\} \end{aligned}$$

$$\begin{tikzcd} X \arrow[d, "f"] & & \mathcal{O}(X) & f^{-1}(A) \in X \\ Y \arrow[r, phantom, "\longmapsto"] & & \mathcal{O}(Y) \arrow[u] & A \subset Y \arrow[u, mapsto] \end{tikzcd}$$

- $\mathsf{CRng}^{\mathrm{op}} \longrightarrow \mathsf{Set}$ sending R to the set of its prime ideals.

- “Topological presheafs”: A presheaf of sets/rings/abelian groups/... over a topological space X is functor $\mathcal{O}(X)^{\mathrm{op}} \longrightarrow \mathsf{Set}$ or $\mathsf{Rng}/\mathsf{AbGp}$.... So $[\mathcal{O}(X)^{\mathrm{op}}, \mathsf{Set}]$, $[\mathcal{O}(X)^{\mathrm{op}}, \mathsf{AbGp}]$ etc. are further examples of categories (of presheaves on a space X). [Don’t confuse this with the “presheaf of open sets” above, that’s something else!]

- The category of affine varieties over k is contravariantly isomorphic to the category of finitely generated reduced k-algebras via $\mathrm{Spec} \colon \mathsf{fgAlg}_k \longrightarrow \mathsf{Aff}_k$.

- There is a global sections functor from $\mathsf{Sh}(X)$ to Set (or AbGp...). If $X = \mathrm{Spec}(A)$ for a commutative ring A, then this global sections functor induces an equivalence between the category of coherent sheaves over X and the category of finitely generated A-modules.

- Given a map $f \colon X \longrightarrow Y$ between topological spaces, we can form so-called “pushforward” or “direct image” and “pullback” or “inverse image” functors between the categories of sheaves:

$$f_* \colon \mathsf{Sh}(X) \longrightarrow \mathsf{Sh}(Y)$$

  with $f_*(\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ is the pushforward functor, and the inverse image functor goes the other way, but its definition is a little more complicated.