Cubical sets axiomatised in a topos

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PSSL 100

Cubical sets

axiomatised in a topos

C. Cohen, T. Coquand, S. Huber & A. Mörtberg
Cubical Type Theory: a constructive interpretation of the univalence axiom (Dec. 2015)

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axiomatise in a topos

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generalise from their particular presheaf topos &

Identity types

$$x \equiv y \qquad [\; x, y : A \;]$$

type of proofs that x equals y

Coquand's axioms for propositional identity types

$\mathrm{refl}\, x \;:\; x \equiv x$

$p : x \equiv y$

$ax_4 \;:\; (x, \mathrm{refl}\, x) \equiv (y, p)$

$\_ \bullet \_ \;:\; x \equiv y \to B x \to B y$

$ax_3 \;:\; (\mathrm{refl}\, x) \cdot b \equiv b$

Coquand's axioms for **propositional** identity types

equivalent to M-L formulation

$$\mathrm{refl}\,x : x \equiv x$$

$$ax_4 : (x, \mathrm{refl}\,x) \equiv (y, p)$$

$$\_\bullet\_ : x \equiv y \to Bx \to By$$

$$ax_3 : (\mathrm{refl}\,x) \cdot b \equiv b$$

EXCEPT THAT COMPUTATION RULE HOLDS UP TO $\equiv$, not definitionally

See Benno Van Den Berg (Apr. 2016) for a category-theoretic analysis

Homotopical view of Equality

$x \equiv y$ $\quad [\, x, y : A \,]$

type of ~~proofs that x equals y~~

[abstract] paths from x to y

Homotopical view of Equality

$x \equiv y$ $\quad [\, x, y : A \,]$

type of ~~proofs that x equals y~~

~~[abstract]~~ paths from x to y

if path = morphism $I \to A$, what does an interval object I in a topos have to satisfy to support the axioms for $\equiv$?

Given

$$\boxed{\begin{tikzcd} 1 \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & I \end{tikzcd}}$$

interval with end-points

in a topos $\mathcal{E}$

for each $A \in \mathcal{E}$ we get

$$\begin{tikzcd} \equiv \arrow[d] \\ A \times A \end{tikzcd} \qquad x \equiv y \stackrel{\text{def}}{=} \{p : A^I \mid p0 = x \wedge p1 = y\}$$

$(x, y : A)$

What's needed for this $\equiv$ to satisfy the axioms?

Coquand's axioms for propositional identity types

$$\mathrm{refl}\ x : x \equiv x$$

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$$\_\bullet\_ : x \equiv y \to B\,x \to B\,y$$

$$ax_3 : (\mathrm{refl}\ x) \cdot b \equiv b$$

Coquand's axioms for propositional identity types

$\mathrm{refl}\ x : \ x \equiv x$

$\mathrm{refl}\ x \overset{\mathrm{def}}{=} \lambda\, i : \mathrm{I}.\ x$

Coquand's axioms for propositional identity types

$$\mathsf{refl}\ x : \ x \equiv x$$

$\lambda i : \mathbb{I}.\, x$ $\qquad$ $p : x \equiv y$

$$ax_4 : \ (x, \mathsf{refl}\ x) \equiv (y, p)$$

Coquand's axioms for propositional identity types

$$\mathrm{refl}\ x : x \equiv x \qquad \lambda i : I.\, x \qquad p : x \equiv y$$

$$ax_4 : (x, \mathrm{refl}\ x) \equiv (y, p)$$

$$ax_4 \stackrel{\mathrm{def}}{=} \lambda i : I.\, (p\, i,\ ?\, i)$$

$$\boxed{\begin{array}{l} ? : I \to (I \to A) \\ ?\,0 = \lambda j.\, x \\ ?\,1 = p \end{array}}$$

Coquand's axioms for propositional identity types

$$\mathrm{refl}\,x : x \equiv x$$

$\lambda i : I.\, x$

$p : x \equiv y$

$$ax_4 : (x, \mathrm{refl}\,x) \equiv (y, p)$$

$$ax_4 \stackrel{\mathrm{def}}{=} \lambda i : I.\, (p\,i,\ ?\,i)$$

$\boxed{\begin{array}{l} \min : I \to I \to I \\ \min 0\, i = 0 \\ \min 1\, i = i \end{array}}$

take
$A = I$
$p = id_I$
$x = 0$
$y = 1$

$\boxed{\begin{array}{l} ? : I \to (I \to A) \\ ?\,0 = \lambda j.\, x \\ ?\,1 = p \end{array}}$

Coquand's axioms for propositional identity types

$\mathrm{refl}\ x : \quad x \equiv x$

$\lambda i : I . x$ $\qquad$ $p : x \equiv y$

$ax_4 \quad : \quad (x, \mathrm{refl}\ x) \equiv (y, p)$

$ax_4 \overset{\mathrm{def}}{=} \lambda i : I . (p\, i, \lambda j : I . p(\min i\, j))$

$$\boxed{\begin{array}{l} \min : I \to I \to I \\ \min 0\, i = 0 \\ \min 1\, i = i \end{array}}$$

If we postulate that I has this "connection" structure, then we can satisfy ax_4.

Coquand's axioms for propositional identity types

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fibre of $\begin{array}{c} B \\ \downarrow \\ A \end{array}$ over $y : A$

Coquand's axioms for propositional identity types

$$\mathrm{refl}\ x : x \equiv x$$

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fibre of $\begin{array}{c} B \\ \downarrow \\ A \end{array}$ over $y : A$

$$ax_3 : (\mathrm{refl}\ x) \cdot b \equiv b$$

Wanted: notion of fibration $\begin{array}{c} B \\ \downarrow \\ A \end{array}$

supporting $\_\cdot\_$ & ax_3, closed under $\Sigma, \Pi, \equiv, \ldots$

C. Cohen, T. Coquand, S. Huber & A. Mörtberg
Cubical Type Theory: a constructive interpretation of the univalence axiom (Dec. 2015)

$\mathcal{E}$ = presheaf topos $\mathbf{Set}^{dM^{op}}$

Lawvere theory of **de Morgan algebra**
||
distributive lattice
+ de Morgan involution

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Cubical Type Theory: a constructive interpretation of the univalence axiom (Dec. 2015)

$\mathcal{E}$ = presheaf topos $\mathbf{Set}^{dM^{op}}$

$\mathbb{I}$ = representable presheaf on the generic de Morgan algebra

0 = least element
1 = greatest elt
min = binary meet

} of generic de M algebra

fibrations = ...

CCHM fibrations

Replace $\mathbf{Set}^{\mathrm{dM}^{op}}$ by an arbitrary topos $\mathcal{E}$ containing a connection algebra $\mathbb{I}$

$$1 \underset{1}{\overset{0}{\rightrightarrows}} \mathbb{I} \underset{\max}{\overset{\min}{\leftleftarrows}} \mathbb{I} \times \mathbb{I}$$

$$\min 0\, i = 0 = \min i\, 0 \qquad \max 1\, i = 1 = \max i\, 1$$

$$\min 1\, i = \max 0\, i = i = \max i\, 0 = \min i\, 1$$

(weaker than a de Morgan algebra)

CCHM fibrations

Replace $\mathbf{Set}^{\mathbf{dM}^{op}}$ by an arbitrary topos $\mathcal{E}$ containing a connection algebra $\mathbb{I}$ and a collection of cofibrant monos classified by $\mathrm{Cof} \rightarrowtail \Omega$ whose properties are to be determined

(CCHM take Cof in $\mathbf{Set}^{\mathbf{dM}^{op}}$ to be given by unions of faces of cubes $\mathbb{I}^n$.)

CCHM fibrations

Replace $\mathrm{Set}^{\mathrm{dM}^{op}}$ by an arbitrary topos $\mathcal{E}$ containing a connection algebra $\mathbb{I}$ and a collection of cofibrant monos classified by $\mathrm{Cof} \rightarrowtail \Omega$ whose properties are to be determined

NOTATION: $\rightarrowtail\!\!\!\!\triangleright$ denotes an $\mathcal{E}$-mono whose classifier factors through $\mathrm{Cof} \rightarrowtail \Omega$.

CCHM fibrations

turn out to be an instance of

Homotopy

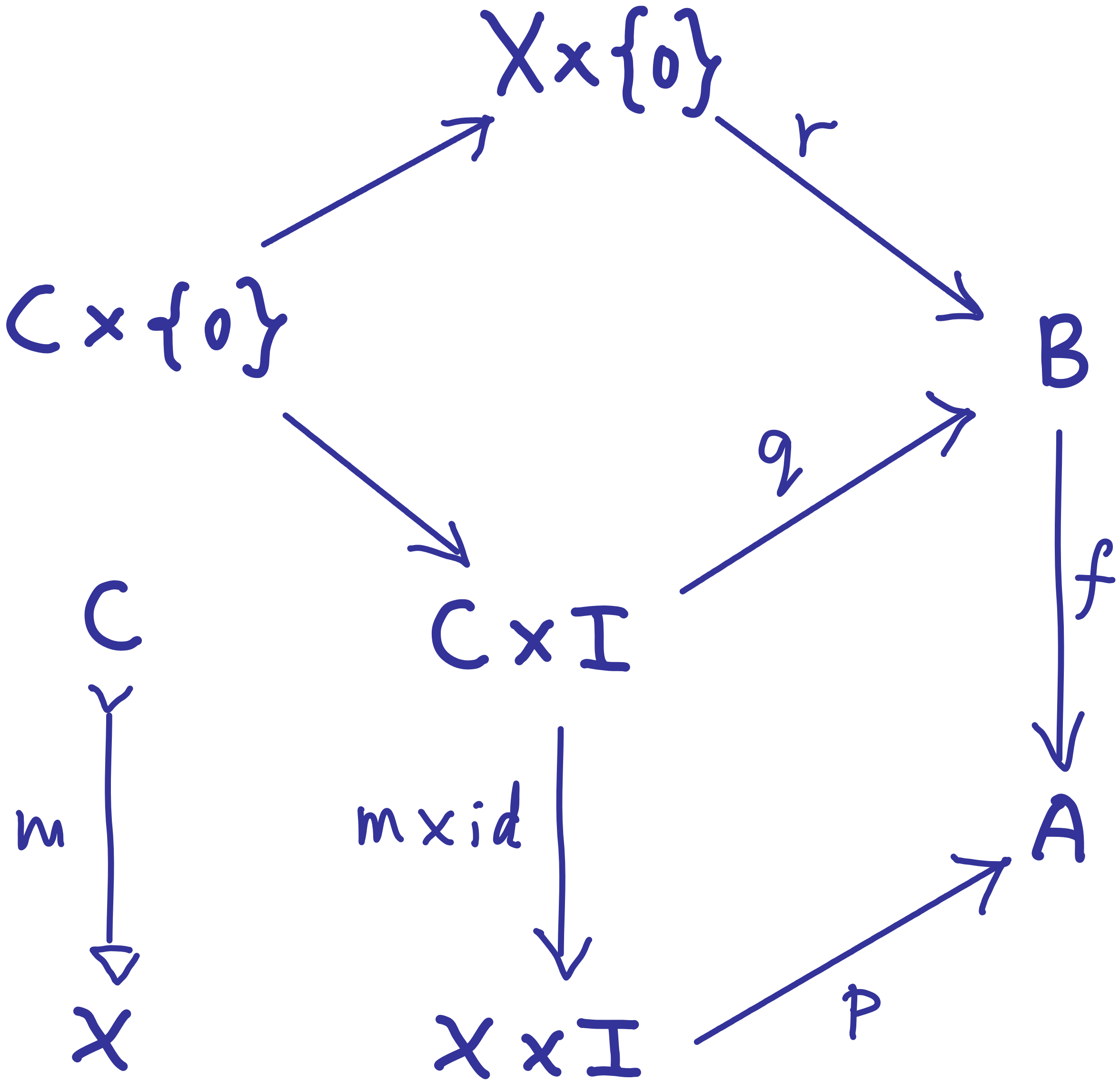
Extension

Lifting

Property

(references ?)

CCHM-fibration structure on $\begin{array}{c} B \\ \downarrow f \\ A \end{array}$



$$\begin{tikzcd}
C \arrow[d, tail, "m"'] \\
X
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X\times\{0\} \arrow[dr, "r"] & \\
C\times\{0\} \arrow[ur] \arrow[dr] & & B \arrow[dd, "f"] \\
 & C\times I \arrow[ur, "q"] \arrow[dd, "m\times id"'] & \\
 & & A \\
 & X\times I \arrow[ur, "p"'] &
\end{tikzcd}$$

every

$p \quad m \quad q \quad r$

CCHM-fibration structure on $\begin{matrix} B \\ \downarrow f \\ A \end{matrix}$

$$
\begin{tikzcd}
 & & X\times\{0\} \arrow[drr, "r"] & & \\
 & C\times\{0\} \arrow[ur] \arrow[dr] & & & B \arrow[dd, "f"] \\
C \arrow[dd, tail, "m"'] & & C\times I \arrow[urr, "q"] \arrow[dd, "m\times id"'] & & \\
 & & & & A \\
X & & X\times I \arrow[urr, "p"] & &
\end{tikzcd}
$$

every

$p \; m \; q \; r$

p is a path in A

CCHM-fibration structure on $\begin{array}{c} B \\ \downarrow f \\ A \end{array}$

$$
\begin{tikzcd}
 & & X \times \{0\} \arrow[dr, "r"] & \\
 & C \times \{0\} \arrow[ur] \arrow[dr] & & B \arrow[dd, "f"] \\
C \arrow[dd, tail, "m"'] & & C \times I \arrow[ur, "q"] \arrow[dd, "m \times id"'] & \\
 & & & A \\
X & & X \times I \arrow[ur, "p"'] &
\end{tikzcd}
$$

every

p m q r

(m, q) is a cofibrant-partial path in B over p

p is a path in A

CCHM-fibration structure on $\begin{array}{c} B \\ \downarrow f \\ A \end{array}$

$$
\begin{tikzcd}
 & & X\times\{0\} \arrow[rr, "r"] & & B \arrow[dd, "f"] \\
 & C\times\{0\} \arrow[ur] \arrow[r] & C\times I \arrow[urr, "q"] \arrow[d, "m\times id"'] & & \\
C \arrow[d, tail, "m"'] & & X\times I \arrow[rr, "p"] & & A \\
X & & & &
\end{tikzcd}
$$

every

p m q r

r extends (m, q) at 0

(m, q) is a cofibrant-partial path in B over p

p is a path in A

CCHM-fibration structure on $\begin{matrix} B \\ \downarrow f \\ A \end{matrix}$

$$\begin{tikzcd}
 & & X\times\{0\} \arrow[dr, "r"] \arrow[dddl, hook, bend right] & \\
C \arrow[dd, "m"'] & C\times I \arrow[dd, "m\times id"'] \arrow[rr, "q"] & & B \arrow[dd, "f"] \\
 & & & \\
X & X\times I \arrow[rr, "p"'] \arrow[uurr, dashed, "\mathrm{fill}(m,q,r)"] & & A
\end{tikzcd}$$

every $p\ m\ q\ r$ has a filling

(& similarly for filling from 1)

CCHM-fibration structure on $\begin{matrix} B \\ \downarrow f \\ A \end{matrix}$

$$
\begin{tikzcd}
 & X\times\{0\} \arrow[dddd, bend right, hook] \arrow[drr, "r"] & & \\
 & & & B \arrow[dd, "f"] \\
C \arrow[dd, rightarrowtail, "m"'] & C\times I \arrow[dd, "m\times id"'] \arrow[urr, "q"] & & \\
 & & & A \\
X & X\times I \arrow[urr, "p"'] \arrow[uuurr, dashed, "\mathrm{fill}(m,q,r)"] & &
\end{tikzcd}
$$

every

$p\ m\ q\ r$

has a filling

(& similarly for filling from 1)

+ naturality in X

CCHM-fibration structure on $\begin{array}{c} B \\ \downarrow f \\ A \end{array}$

$$
\begin{tikzcd}
 & & X\times\{0\} \arrow[dr, "r"] \arrow[ddddl, bend right, hook] & \\
 & & & B \arrow[dd, "f"] \\
C \arrow[dd, tail, "m"'] & C\times I \arrow[urr, "q"] \arrow[dd, "m\times id"'] & & \\
 & & & A \\
X & X\times I \arrow[urr, "p"'] \arrow[uuurr, dashed, "\mathrm{fill}(m,q,r)"] & &
\end{tikzcd}
$$

every $p\ m\ q\ r$ has a filling

(& similarly for filling from 1)

~~+ naturality in X~~

↓ internalize

"Composition" structure on $\begin{array}{c} B \\ \downarrow f \\ A \end{array}$

$$
\begin{tikzcd}
 & & X\times\{0\} \arrow[dr, "r"] & \\
 & C\times\{0\} \arrow[ur] \arrow[dr] & & B \arrow[dd, "f"] \\
C \arrow[d, tail, "m"'] & & C\times I \arrow[ur, "q"] \arrow[d, "m\times id"'] & \\
X & & X\times I \arrow[r, "p"] & A
\end{tikzcd}
$$

every

p m q r

"Composition" structure on $\begin{matrix} B \\ \downarrow f \\ A \end{matrix}$

$$\begin{tikzcd}
 & & X\times\{1\} \arrow[dr, dashed, "\mathrm{Comp}(p,m,q,r)"] & \\
 & C\times\{1\} \arrow[ur] \arrow[d] & & B \arrow[d, "f"] \\
 C \arrow[d, tail, "m"'] & C\times I \arrow[d, "m\times\mathrm{id}"'] \arrow[urr, "q"] & & A \\
 X & X\times I \arrow[urr, "p"'] & &
\end{tikzcd}$$

every $p\ m\ q\ r$ extends at 1

(& similarly, interchanging 0 & 1)

Theorem If $\mathrm{Cof} \rightarrowtail \Omega$ satisfies

$\mathrm{false} \in \mathrm{Cof}$	$\forall i : \mathbb{I}, \varphi : \Omega.\ \varphi \in \mathrm{Cof} \Rightarrow \varphi \vee i = b \in \mathrm{Cof}$ (for $b = 0, 1$)

then $f : B \to A$ is a CCHM-fibration if it has a composition structure.

Theorem If $\mathrm{Cof} \rightarrowtail \Omega$ satisfies

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then $f : B \to A$ is a CCHM-fibration if it has a composition structure.

Corollary CCHM-fibrations are closed under taking Π & Σ and have prop^{l} id types given by $\mathbb{I}$-paths.

Theorem If $\mathrm{Cof} \rightarrowtail \Omega$ satisfies

$\mathrm{false} \in \mathrm{Cof}$	$\forall i : \mathbb{I}, \varphi : \Omega .\ \varphi \in \mathrm{Cof} \Rightarrow \varphi \vee i = b \in \mathrm{Cof}$ (for $b = 0, 1$)

then $f : B \to A$ is a CCHM-fibration if it has a composition structure.

Theorem If $\mathbb{I}$ satisfies

$$\forall P : \Omega^{\mathbb{I}} .\ (\forall i .\ P i \vee \neg P i) \Rightarrow (\forall i . P i) \vee (\forall i . \neg P i)$$

then CCHM-fibrations are closed under $+$ & contain $\mathcal{E}$'s natural number object.

For proofs, see:

Ian Orton & AMP, Axioms for Modelling Cubical Type Theory in a Topos (Apr. 2016)

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in the internal language of a category-with-families associated with the topos

(∃ Agda formalization)

Satisfying the axioms

$\mathbf{Set}^{\mathbb{C}^{op}}$ for any $\mathbb{C}$ which

- contains a path connection algebra
- is co-sifted (= inhabited & all span categories are connected)

E.g. $\mathbb{C} = \mathrm{dM}$, $\mathbb{C} = \Delta$

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Hofmann-Streicher style universe of fibrations.
Univalence via "glueing" construct in $\mathrm{Set}^{\mathrm{dm}^{\mathrm{op}}}$

universe of types in which
equivalence of types up to path-equality
is equivalent to
path-equality of types

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Univalence via "glueing" construct in $\mathbf{Set}^{\mathrm{dm}^{\mathrm{op}}}$

Orton + AMP: $\left\{ \begin{array}{l} \checkmark \text{ axiomatic "glueing" + weak form of univalence} \\ \\ \times \text{ CCHM universe} \end{array} \right.$

Wanted

Examples of type-theoretic universes that are self-evidentally univalent (& not truncated).