Categories of spaces built from local models

Zhen Lin Low

Department of Pure Mathematics and Mathematical Statistics
University of Cambridge

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- What are the properties of these categories of spaces built from local models?

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- An **admissible ecumene** consists of data as above, satisfying some further technical conditions.

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- In examples of interest, the category of spaces built from local models is the universal effective admissible ecumene into which the category of local models embeds.

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- Every finite diagram in $\mathcal{C}$ has an E-weak limit.

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$$\begin{aligned} \mathrm{Hom}(F_! A, B) &\to \mathrm{Hom}(A, F^* B) \\ h &\mapsto F^* h \circ \sigma_A \end{aligned}$$

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Remark. This is most of the hard work in showing that the 2-subcategory of effective admissible ecumenae is bireflective in the 2-category of admissible ecumenae.

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