A Seifert–van Kampen theorem in non-abelian algebra

Tim Van der Linden
joint work with Mathieu Duckerts-Antoine

Fonds de la Recherche Scientifique–FNRS
Université catholique de Louvain

100th Peripatetic Seminar on Sheaves and Logic
22nd of May 2016 | Cambridge

Introduction: categorical tools for homological algebra

- Concrete aim: understanding (co)homology of groups
- Which aspects of group cohomology are typical for groups, and which function for purely formal reasons, so that a categorical argument suffices to understand and apply these in other settings?
- Conversely, what do our "homological needs" tell us about categories of non-abelian algebraic structures?

Today: towards an algebraic version of the *Seifert–van Kampen* theorem

Introduction: categorical tools for homological algebra

- Concrete aim: understanding (co)homology of groups
- Which aspects of group cohomology are typical for groups, and which function for purely formal reasons, so that a categorical argument suffices to understand and apply these in other settings?
- Conversely, what do our "homological needs" tell us about categories of non-abelian algebraic structures?

Today: towards an algebraic version of the *Seifert–van Kampen* theorem

Introduction: categorical tools for homological algebra

- Concrete aim: understanding (co)homology of groups
- Which aspects of group cohomology are typical for groups, and which function for purely formal reasons, so that a categorical argument suffices to understand and apply these in other settings?
- Conversely, what do our "homological needs" tell us about categories of non-abelian algebraic structures?

Today: towards an algebraic version of the *Seifert–van Kampen* theorem

Introduction: categorical tools for homological algebra

- Concrete aim: understanding (co)homology of groups
- Which aspects of group cohomology are typical for groups, and which function for purely formal reasons, so that a categorical argument suffices to understand and apply these in other settings?
- Conversely, what do our “homological needs” tell us about categories of non-abelian algebraic structures?

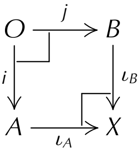
Today: towards an algebraic version of the *Seifert–van Kampen* theorem

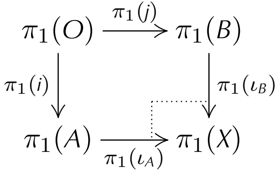
Introduction: categorical tools for homological algebra

- Concrete aim: understanding (co)homology of groups
- Which aspects of group cohomology are typical for groups, and which function for purely formal reasons, so that a categorical argument suffices to understand and apply these in other settings?
- Conversely, what do our "homological needs" tell us about categories of non-abelian algebraic structures?

Today: towards an algebraic version of the *Seifert–van Kampen* theorem

Consider a square of pointed topological spaces as on the left,





where

- A, B and O are open, path-connected subspaces of X;
- $X = A +_O B = A \cup B$: A and B cover X, the square is a pushout;
- $O = A \times_X B = A \cap B$: the square is a pullback.

Then the square on the right is a pushout in Gp.

- Many variations on this theme exist, in topology, in algebraic geometry and in algebra.
- Grothendieck's version in [SGA1, 1971] is based on descent theory.

The classical Seifert–van Kampen theorem [1931, 1933]

Consider a square of pointed topological spaces as on the left,

$$\begin{array}{ccc} O & \xrightarrow{j} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \iota_B} \\ A & \xrightarrow[\iota_A]{} & X \end{array} \qquad\qquad \begin{array}{ccc} \pi_1(O) & \xrightarrow{\pi_1(j)} & \pi_1(B) \\ {\scriptstyle \pi_1(i)}\downarrow & & \downarrow{\scriptstyle \pi_1(\iota_B)} \\ \pi_1(A) & \xrightarrow[\pi_1(\iota_A)]{} & \pi_1(X) \end{array}$$

where

- A, B and O are open, path-connected subspaces of X;
- $X = A +_O B = A \cup B$: A and B cover X, the square is a pushout;
- $O = A \times_X B = A \cap B$: the square is a pullback.

Then the square on the right is a pushout in Gp.

- Many variations on this theme exist, in topology, in algebraic geometry and in algebra.
- Grothendieck's version in [SGA1, 1971] is based on descent theory.

The classical Seifert–van Kampen theorem [1931, 1933]

Consider a square of pointed topological spaces as on the left,

$$
\begin{array}{ccc}
O & \xrightarrow{\;j\;} & B \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \iota_B} \\
A & \xrightarrow[\;\iota_A\;]{} & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
\pi_1(O) & \xrightarrow{\;\pi_1(j)\;} & \pi_1(B) \\
{\scriptstyle \pi_1(i)}\downarrow & & \downarrow{\scriptstyle \pi_1(\iota_B)} \\
\pi_1(A) & \xrightarrow[\;\pi_1(\iota_A)\;]{} & \pi_1(X)
\end{array}
$$

where

- A, B and O are open, path-connected subspaces of X;
- $X = A +_O B = A \cup B$: A and B cover X, the square is a pushout;
- $O = A \times_X B = A \cap B$: the square is a pullback.

Then the square on the right is a pushout in Gp.

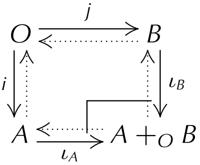
- Many variations on this theme exist, in topology,
  in algebraic geometry and in algebra.
- Grothendieck's version in [SGA1, 1971] is based on descent theory.

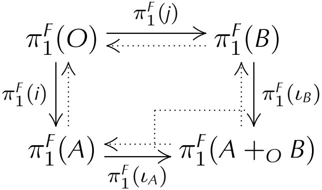
A Seifert–van Kampen theorem in non-abelian algebra

Let $F\colon \mathbb{C} \to \mathbb{X}$ be a functor where

- $\mathbb{C}$ is semi-abelian algebraically coherent with enough projectives;
- $\mathbb{X}$ is (almost) abelian;
- F is a regular epi–reflector which
  preserves pullbacks of regular epis along split epis

and consider in $\mathbb{C}$ a pushout of split monomorphisms as on the left.





Then the square on the right is a pushout square in $\mathbb{X}$.

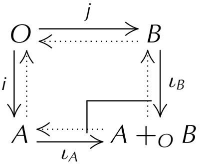
- Any reflector from a *category of interest* to an abelian subvariety.
- Our proof technique is new, based on categorical Galois theory, which is related to descent—no obvious connection with [SGA1].
- This is a first step: perhaps something more general is possible.

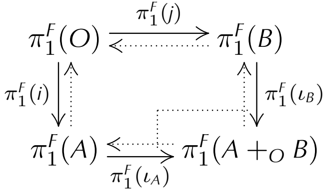
A Seifert–van Kampen theorem in non-abelian algebra

Let $F\colon \mathbb{C} \to \mathbb{X}$ be a functor where

- $\mathbb{C}$ is semi-abelian algebraically coherent with enough projectives;
- $\mathbb{X}$ is (almost) abelian;
- F is a regular epi–reflector which
  preserves pullbacks of regular epis along split epis

and consider in $\mathbb{C}$ a pushout of split monomorphisms as on the left.





Then the square on the right is a pushout square in $\mathbb{X}$.

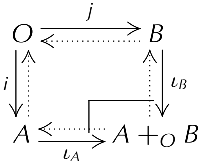
- Any reflector from a *category of interest* to an abelian subvariety.
- Our proof technique is new, based on categorical Galois theory, which is related to descent—no obvious connection with [SGA1].
- This is a first step: perhaps something more general is possible.

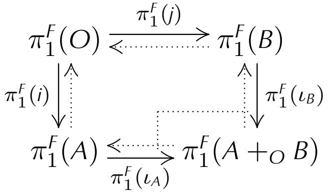
A Seifert–van Kampen theorem in non-abelian algebra

Let $F\colon \mathbb{C} \to \mathbb{X}$ be a functor where

- $\mathbb{C}$ is semi-abelian algebraically coherent with enough projectives;
- $\mathbb{X}$ is (almost) abelian;
- F is a regular epi–reflector which
  preserves pullbacks of regular epis along split epis

and consider in $\mathbb{C}$ a pushout of split monomorphisms as on the left.





Then the square on the right is a pushout square in $\mathbb{X}$.

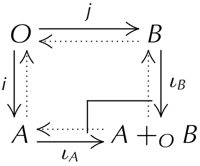
- Any reflector from a *category of interest* to an abelian subvariety.
- Our proof technique is new, based on categorical Galois theory, which is related to descent—no obvious connection with [SGA1].
- This is a first step: perhaps something more general is possible.

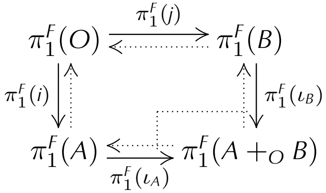
A Seifert–van Kampen theorem in non-abelian algebra

Let $F\colon \mathbb{C} \to \mathbb{X}$ be a functor where

- $\mathbb{C}$ is semi-abelian algebraically coherent with enough projectives;
- $\mathbb{X}$ is (almost) abelian;
- F is a regular epi–reflector which
  preserves pullbacks of regular epis along split epis

and consider in $\mathbb{C}$ a pushout of split monomorphisms as on the left.





Then the square on the right is a pushout square in $\mathbb{X}$.

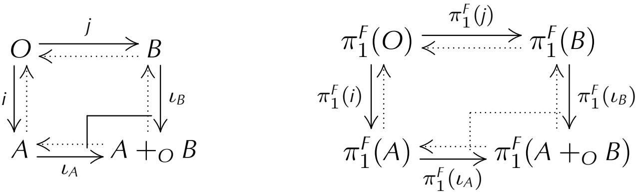
- Any reflector from a *category of interest* to an abelian subvariety.
- Our proof technique is new, based on categorical Galois theory, which is related to descent—no obvious connection with [SGA1].
- This is a first step: perhaps something more general is possible.

A Seifert–van Kampen theorem in non-abelian algebra

Let $F\colon \mathbb{C} \to \mathbb{X}$ be a functor where

- $\mathbb{C}$ is semi-abelian algebraically coherent with enough projectives;
- $\mathbb{X}$ is (almost) abelian;
- F is a regular epi–reflector which
  preserves pullbacks of regular epis along split epis

and consider in $\mathbb{C}$ a pushout of split monomorphisms as on the left.



Then the square on the right is a pushout square in $\mathbb{X}$.

- Any reflector from a *category of interest* to an abelian subvariety.
- Our proof technique is new, based on categorical Galois theory, which is related to descent—no obvious connection with [SGA1].
- This is a first step: perhaps something more general is possible.

The semi-abelian context

Category is	Noteworthy aspects	Examples
abelian	classical homological algebra additive structure on homsets subobjects are normal	Ab, Mod_R, $\mathsf{Sh}_{\mathscr{T}}(\mathsf{Ab})$
semi-abelian + (LACC)	internal (co)homology	Gp, XMod, Lie_K, $\mathsf{HopfAlg}_{K,\mathrm{coc}}$
semi-abelian + algebraically coherent	Seifert–van Kampen universal central extensions three subobjects lemma	$\mathsf{Nil}_n(\mathsf{Gp})$, $\mathsf{Gp}_{\mathrm{tf}}$, $\mathsf{Cat}^n(\mathsf{Gp})$ all *categories of interest*: Rng, Alg_K, Leib_K
semi-abelian + (SH)	internal crossed modules action on abelian object is Beck module cohomology via higher centrality	NAAlg_K, Jordan algebras, Heyting semilattices
semi-abelian	basic homological algebra internal actions	Loop, DiGp all *varieties of* Ω*-groups*

- **semi-abelian** = pointed + Barr exact + protomodular + finite sums
- Internal actions correspond to split extensions: if B acts on X via ξ,

$$0 \longrightarrow X \rightarrowtail X \rtimes_{\xi} B \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0.$$

  Thus an action corresponds to a **point**: a couple (f, s) with $fs = 1_B$. Think of a point as a "non-abelian module".

- commutators measure non-abelianness $\rightsquigarrow$ commutator conditions

The semi-abelian context

Category is	Noteworthy aspects	Examples
abelian	classical homological algebra additive structure on homsets subobjects are normal	Ab, Mod_R, $\mathrm{Sh}_{\mathscr{T}}(\mathrm{Ab})$
semi-abelian + (LACC)	internal (co)homology	Gp, XMod, Lie_K, $\mathrm{HopfAlg}_{K,\mathrm{coc}}$
semi-abelian + algebraically coherent	Seifert–van Kampen universal central extensions three subobjects lemma	$\mathrm{Nil}_n(\mathrm{Gp})$, $\mathrm{Gp}_{\mathrm{tf}}$, $\mathrm{Cat}^n(\mathrm{Gp})$ all *categories of interest*: Rng, Alg_K, Leib_K
semi-abelian + (SH)	internal crossed modules action on abelian object is Beck module cohomology via higher centrality	NAAlg_K, Jordan algebras, Heyting semilattices
semi-abelian	basic homological algebra internal actions	Loop, DiGp all *varieties of* Ω*-groups*

- **semi-abelian** = pointed + Barr exact + protomodular + finite sums
- Internal actions correspond to split extensions: if B acts on X via ξ,

$$0 \longrightarrow X \rightarrowtail X \rtimes_{\xi} B \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0.$$

  Thus an action corresponds to a **point**: a couple (f, s) with $fs = 1_B$. Think of a point as a “non-abelian module”.

- commutators measure non-abelianness $\rightsquigarrow$ commutator conditions

The semi-abelian context

[G Janelidze, L Márki & W Tholen, 2002]

Category is	Noteworthy aspects	Examples
abelian	classical homological algebra additive structure on homsets subobjects are normal	Ab, Mod_R, $\mathsf{Sh}_{\mathscr{T}}(\mathsf{Ab})$
semi-abelian + (LACC)	internal (co)homology	Gp, XMod, Lie_K, $\mathsf{HopfAlg}_{K,\mathrm{coc}}$
semi-abelian + algebraically coherent	Seifert–van Kampen universal central extensions three subobjects lemma	$\mathsf{Nil}_n(\mathsf{Gp})$, $\mathsf{Gp}_{\mathrm{tf}}$, $\mathsf{Cat}^n(\mathsf{Gp})$ all *categories of interest*: Rng, Alg_K, Leib_K
semi-abelian + (SH)	internal crossed modules action on abelian object is Beck module cohomology via higher centrality	NAAlg_K, Jordan algebras, Heyting semilattices
semi-abelian	basic homological algebra internal actions	Loop, DiGp all *varieties of* Ω*-groups*

- **semi-abelian** = pointed + Barr exact + protomodular + finite sums
- Internal actions correspond to split extensions: if B acts on X via ξ,

$$0 \longrightarrow X \rightarrowtriangle X \rtimes_\xi B \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0.$$

  Thus an action corresponds to a **point**: a couple (f, s) with $fs = 1_B$.
  Think of a point as a "non-abelian module".
- commutators measure non-abelianness $\rightsquigarrow$ commutator conditions

The semi-abelian context

[D Bourn, 1991]

Category is	Noteworthy aspects	Examples
abelian	classical homological algebra additive structure on homsets subobjects are normal	Ab, Mod_R, $\mathrm{Sh}_{\mathscr{T}}(\mathrm{Ab})$
semi-abelian + (LACC)	internal (co)homology	Gp, XMod, Lie_K, $\mathrm{HopfAlg}_{K,\mathrm{coc}}$
semi-abelian + algebraically coherent	Seifert–van Kampen universal central extensions three subobjects lemma	$\mathrm{Nil}_n(\mathrm{Gp})$, $\mathrm{Gp}_{\mathrm{tf}}$, $\mathrm{Cat}^n(\mathrm{Gp})$ all *categories of interest*: Rng, Alg_K, Leib_K
semi-abelian + (SH)	internal crossed modules action on abelian object is Beck module cohomology via higher centrality	NAAlg_K, Jordan algebras, Heyting semilattices
semi-abelian	basic homological algebra internal actions	Loop, DiGp all *varieties of* Ω*-groups*

- **semi-abelian** = pointed + Barr exact + **protomodular** + finite sums
- Internal actions correspond to split extensions: if B acts on X via ξ,

$$0 \longrightarrow X \rightarrowtail X \rtimes_\xi B \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0.$$

  Thus an action corresponds to a **point**: a couple (f, s) with $fs = 1_B$.
  Think of a point as a "non-abelian module".
- commutators measure non-abelianness $\rightsquigarrow$ commutator conditions

The semi-abelian context

[D Bourn & G Janelidze, 1998]

Category is	Noteworthy aspects	Examples
abelian	classical homological algebra additive structure on homsets subobjects are normal	Ab, Mod_R, $\mathrm{Sh}_{\mathscr{T}}(\mathrm{Ab})$
semi-abelian + (LACC)	internal (co)homology	Gp, XMod, Lie_K, $\mathrm{HopfAlg}_{K,\mathrm{coc}}$
semi-abelian + algebraically coherent	Seifert–van Kampen universal central extensions three subobjects lemma	$\mathrm{Nil}_n(\mathrm{Gp})$, $\mathrm{Gp}_{\mathrm{tf}}$, $\mathrm{Cat}^n(\mathrm{Gp})$ all *categories of interest*: Rng, Alg_K, Leib_K
semi-abelian + (SH)	internal crossed modules action on abelian object is Beck module cohomology via higher centrality	NAAlg_K, Jordan algebras, Heyting semilattices
semi-abelian	basic homological algebra **internal actions**	Loop, DiGp all *varieties of Ω-groups*

- **semi-abelian** = pointed + Barr exact + protomodular + finite sums
- Internal actions correspond to split extensions: if B acts on X via ξ,

$$0 \longrightarrow X \rhd\!\!\longrightarrow X \rtimes_{\xi} B \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0.$$

  Thus an action corresponds to a **point**: a couple (f, s) with $fs = 1_B$.
  Think of a point as a "non-abelian module".
- commutators measure non-abelianness $\rightsquigarrow$ commutator conditions

The semi-abelian context [N Martins–Ferreira & TVdL, 2012]

Category is	Noteworthy aspects	Examples
abelian	classical homological algebra additive structure on homsets subobjects are normal	Ab, Mod_R, $\mathrm{Sh}_{\mathscr{T}}(\mathrm{Ab})$
semi-abelian + (LACC)	internal (co)homology	Gp, XMod, Lie_K, $\mathrm{HopfAlg}_{K,\mathrm{coc}}$
semi-abelian + algebraically coherent	Seifert–van Kampen universal central extensions three subobjects lemma	$\mathrm{Nil}_n(\mathrm{Gp})$, $\mathrm{Gp}_{\mathrm{tf}}$, $\mathrm{Cat}^n(\mathrm{Gp})$ all *categories of interest*: Rng, Alg_K, Leib_K
semi-abelian + (SH)	**internal crossed modules** action on abelian object is Beck module cohomology via higher centrality	NAAlg_K, Jordan algebras, Heyting semilattices
semi-abelian	basic homological algebra internal actions	Loop, DiGp all *varieties of* Ω*-groups*

- **semi-abelian** = pointed + Barr exact + protomodular + finite sums
- Internal actions correspond to split extensions: if B acts on X via ξ,

$$0 \longrightarrow X \rightarrowtail X \rtimes_\xi B \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0.$$

  Thus an action corresponds to a **point**: a couple (f, s) with $fs = 1_B$.
  Think of a point as a "non-abelian module".
- commutators measure non-abelianness $\rightsquigarrow$ commutator conditions

The semi-abelian context [D Bourn & G Janelidze, 2004]

Category is	Noteworthy aspects	Examples
abelian	classical homological algebra additive structure on homsets subobjects are normal	Ab, Mod_R, $\mathsf{Sh}_{\mathscr{T}}(\mathsf{Ab})$
semi-abelian + (LACC)	internal (co)homology	Gp, XMod, Lie_K, $\mathsf{HopfAlg}_{K,\mathrm{coc}}$
semi-abelian + algebraically coherent	Seifert–van Kampen universal central extensions three subobjects lemma	$\mathsf{Nil}_n(\mathsf{Gp})$, $\mathsf{Gp}_{\mathrm{tf}}$, $\mathsf{Cat}^n(\mathsf{Gp})$ all *categories of interest*: Rng, Alg_K, Leib_K
semi-abelian + (SH)	internal crossed modules **action on abelian object is Beck module** cohomology via higher centrality	NAAlg_K, Jordan algebras, Heyting semilattices
semi-abelian	basic homological algebra internal actions	Loop, DiGp all *varieties of Ω-groups*

- **semi-abelian** = pointed + Barr exact + protomodular + finite sums
- Internal actions correspond to split extensions: if B acts on X via ξ,

$$0 \longrightarrow X \rightarrowtail X \rtimes_{\xi} B \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0.$$

  Thus an action corresponds to a **point**: a couple (f, s) with $fs = 1_B$. Think of a point as a "non-abelian module".
- commutators measure non-abelianness $\rightsquigarrow$ commutator conditions

The semi-abelian context

[D Rodelo & TVdL, 2012, 2016]

Category is	Noteworthy aspects	Examples
abelian	classical homological algebra additive structure on homsets subobjects are normal	Ab, Mod_R, $\mathsf{Sh}_{\mathscr{T}}(\mathsf{Ab})$
semi-abelian + (LACC)	internal (co)homology	Gp, XMod, Lie_K, $\mathsf{HopfAlg}_{K,\mathrm{coc}}$
semi-abelian + algebraically coherent	Seifert–van Kampen universal central extensions three subobjects lemma	$\mathsf{Nil}_n(\mathsf{Gp})$, $\mathsf{Gp}_{\mathrm{tf}}$, $\mathsf{Cat}^n(\mathsf{Gp})$ all *categories of interest*: Rng, Alg_K, Leib_K
semi-abelian + (SH)	internal crossed modules action on abelian object is Beck module **cohomology via higher centrality**	NAAlg_K, Jordan algebras, Heyting semilattices
semi-abelian	basic homological algebra internal actions	Loop, DiGp all *varieties of Ω-groups*

- **semi-abelian** = pointed + Barr exact + protomodular + finite sums
- Internal actions correspond to split extensions: if B acts on X via ξ,

$$0 \longrightarrow X \rightarrowtail X \rtimes_{\xi} B \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0.$$

  Thus an action corresponds to a **point**: a couple (f, s) with $fs = 1_B$.
  Think of a point as a "non-abelian module".
- commutators measure non-abelianness $\rightsquigarrow$ commutator conditions

The semi-abelian context

[JRA Gray, 2010]

Category is	Noteworthy aspects	Examples
abelian	classical homological algebra additive structure on homsets subobjects are normal	Ab, Mod_R, $\mathrm{Sh}_{\mathscr{T}}(\mathrm{Ab})$
semi-abelian + (LACC)	internal (co)homology	Gp, XMod, Lie_K, $\mathrm{HopfAlg}_{K,\mathrm{coc}}$
semi-abelian + algebraically coherent	Seifert–van Kampen universal central extensions three subobjects lemma	$\mathrm{Nil}_n(\mathrm{Gp})$, $\mathrm{Gp}_{\mathrm{tf}}$, $\mathrm{Cat}^n(\mathrm{Gp})$ all *categories of interest*: Rng, Alg_K, Leib_K
semi-abelian + (SH)	internal crossed modules action on abelian object is Beck module cohomology via higher centrality	NAAlg_K, Jordan algebras, Heyting semilattices
semi-abelian	basic homological algebra internal actions	Loop, DiGp all *varieties of* Ω*-groups*

- **semi-abelian** = pointed + Barr exact + protomodular + finite sums
- Internal actions correspond to split extensions: if B acts on X via ξ,

$$0 \longrightarrow X \rightarrowtail X \rtimes_{\xi} B \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0.$$

  Thus an action corresponds to a **point**: a couple (f, s) with $fs = 1_B$. Think of a point as a "non-abelian module".

- commutators measure non-abelianness $\rightsquigarrow$ commutator conditions

The semi-abelian context

[AS Cigoli, JRA Gray & TVdL, 2015]

Category is	Noteworthy aspects	Examples
abelian	classical homological algebra additive structure on homsets subobjects are normal	Ab, Mod_R, $\mathsf{Sh}_{\mathscr{T}}(\mathsf{Ab})$
semi-abelian + (LACC)	internal (co)homology	Gp, XMod, Lie_K, $\mathsf{HopfAlg}_{K,\mathrm{coc}}$
semi-abelian + algebraically coherent	Seifert–van Kampen universal central extensions three subobjects lemma	$\mathsf{Nil}_n(\mathsf{Gp})$, $\mathsf{Gp}_{\mathrm{tf}}$, $\mathsf{Cat}^n(\mathsf{Gp})$ all *categories of interest*: Rng, Alg_K, Leib_K
semi-abelian + (SH)	internal crossed modules action on abelian object is Beck module cohomology via higher centrality	NAAlg_K, Jordan algebras, Heyting semilattices
semi-abelian	basic homological algebra internal actions	Loop, DiGp all *varieties of* Ω*-groups*

- **semi-abelian** = pointed + Barr exact + protomodular + finite sums
- Internal actions correspond to split extensions: if B acts on X via ξ,

$$0 \longrightarrow X \rightarrowtail X \rtimes_\xi B \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0.$$

  Thus an action corresponds to a **point**: a couple (f, s) with $fs = 1_B$.
  Think of a point as a "non-abelian module".
- commutators measure non-abelianness $\rightsquigarrow$ commutator conditions

The semi-abelian context

Category is	Noteworthy aspects	Examples
abelian	classical homological algebra additive structure on homsets subobjects are normal	Ab, Mod_R, $\mathrm{Sh}_{\mathscr{T}}(\mathrm{Ab})$
semi-abelian + (LACC)	internal (co)homology	Gp, XMod, Lie_K, $\mathrm{HopfAlg}_{K,\mathrm{coc}}$
semi-abelian + algebraically coherent	Seifert–van Kampen universal central extensions three subobjects lemma	$\mathrm{Nil}_n(\mathrm{Gp})$, $\mathrm{Gp}_{\mathrm{tf}}$, $\mathrm{Cat}^n(\mathrm{Gp})$ all *categories of interest*: Rng, Alg_K, Leib_K
semi-abelian + (SH)	internal crossed modules action on abelian object is Beck module cohomology via higher centrality	NAAlg_K, Jordan algebras, Heyting semilattices
semi-abelian	basic homological algebra internal actions	Loop, DiGp all *varieties of Ω-groups*

- **semi-abelian** = pointed + Barr exact + protomodular + finite sums
- Internal actions correspond to split extensions: if B acts on X via ξ,

$$0 \longrightarrow X \rightarrowtail X \rtimes_\xi B \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0.$$

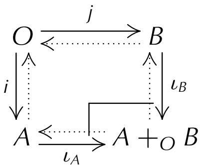
  Thus an action corresponds to a **point**: a couple (f, s) with $fs = 1_B$.
  Think of a point as a "non-abelian module".
- commutators measure non-abelianness $\rightsquigarrow$ commutator conditions

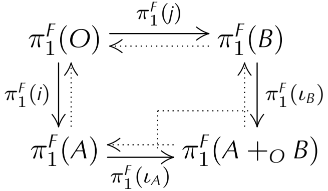
A Seifert–van Kampen theorem in non-abelian algebra

Let $F\colon \mathbb{C} \to \mathbb{X}$ be a functor where

- $\mathbb{C}$ is semi-abelian algebraically coherent with enough projectives;
- $\mathbb{X}$ is (almost) abelian;
- F is a regular epi–reflector which
  preserves pullbacks of regular epis along split epis

and consider in $\mathbb{C}$ a pushout of split monomorphisms as on the left.





Then the square on the right is a pushout square in $\mathbb{X}$.

The fundamental group, I

[G Janelidze & GM Kelly, 1994]

Categorical Galois theory provides notions such as *central extensions* and *fundamental groups* relative to a suitable adjunction $F\colon \mathbb{C} \to \mathbb{X}$.

When $\mathbb{C}$ is semi-abelian and F is a regular epi–reflector which preserves pullbacks of regular epimorphisms along split epimorphisms, there is a derived reflection $F_1\colon \mathrm{Ext}(\mathbb{C}) \to \mathrm{NExt}(\mathbb{C})$.

- $\mathrm{NExt}(\mathbb{C})$ is the category of **normal extensions** relative to F.
- If $F = \mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$, then $F_1\colon \mathrm{Ext}(\mathrm{Gp}) \to \mathrm{CExt}(\mathrm{Gp})$ is the **centralisation functor** sending $f\colon X \to Y$ to $F_1(f)\colon \frac{X}{[X,\mathrm{Ker}(f)]} \to Y$.

The **fundamental group functor** of F is the right Kan extension π_1^F as in

$$\begin{array}{ccc}
\mathrm{Ext}(\mathbb{C}) & \xrightarrow{F_1} & \mathrm{NExt}(\mathbb{C}) \\
{\scriptstyle \mathrm{Cod}}\downarrow & \nearrow & \downarrow{\scriptstyle \mathrm{Ker}} \\
\mathbb{C} & \underset{\pi_1^F}{\dashrightarrow} & \mathbb{X}.
\end{array}$$

If $\mathbb{C}$ has enough projectives, and $\mathbb{X}$ is closed in $\mathbb{C}$ under regular quotients, then $\pi_1^F \cong \mathrm{L}_1 F$, the first simplicial left derived functor of F.

The fundamental group, I

Categorical Galois theory provides notions such as *central extensions* and *fundamental groups* relative to a suitable adjunction $F\colon \mathbb{C} \to \mathbb{X}$.

When $\mathbb{C}$ is semi-abelian and F is a regular epi–reflector which preserves pullbacks of regular epimorphisms along split epimorphisms, there is a derived reflection $F_1\colon \mathrm{Ext}(\mathbb{C}) \to \mathrm{NExt}(\mathbb{C})$.

- $\mathrm{NExt}(\mathbb{C})$ is the category of **normal extensions** relative to F.
- If $F = \mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$, then $F_1\colon \mathrm{Ext}(\mathrm{Gp}) \to \mathrm{CExt}(\mathrm{Gp})$ is the **centralisation functor** sending $f\colon X \to Y$ to $F_1(f)\colon \frac{X}{[X,\mathrm{Ker}(f)]} \to Y$.

The **fundamental group functor** of F is the right Kan extension π_1^F as in

$$\begin{array}{ccc} \mathrm{Ext}(\mathbb{C}) & \xrightarrow{F_1} & \mathrm{NExt}(\mathbb{C}) \\ {\scriptstyle \mathrm{Cod}}\downarrow & \nearrow & \downarrow{\scriptstyle \mathrm{Ker}} \\ \mathbb{C} & \underset{\pi_1^F}{\dashrightarrow} & \mathbb{X}. \end{array}$$

If $\mathbb{C}$ has enough projectives, and $\mathbb{X}$ is closed in $\mathbb{C}$ under regular quotients, then $\pi_1^F \cong \mathrm{L}_1 F$, the first simplicial left derived functor of F.

The fundamental group, I

Categorical Galois theory provides notions such as *central extensions* and *fundamental groups* relative to a suitable adjunction $F\colon \mathbb{C} \to \mathbb{X}$.

When $\mathbb{C}$ is semi-abelian and F is a regular epi–reflector which preserves pullbacks of regular epimorphisms along split epimorphisms, there is a derived reflection $F_1\colon \mathrm{Ext}(\mathbb{C}) \to \mathrm{NExt}(\mathbb{C})$.

- $\mathrm{NExt}(\mathbb{C})$ is the category of **normal extensions** relative to F.
- If $F = \mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$, then $F_1\colon \mathrm{Ext}(\mathrm{Gp}) \to \mathrm{CExt}(\mathrm{Gp})$ is the **centralisation functor** sending $f\colon X \to Y$ to $F_1(f)\colon \frac{X}{[X,\mathrm{Ker}(f)]} \to Y$.

The **fundamental group functor** of F is the right Kan extension π_1^F as in

$$\begin{array}{ccc} \mathrm{Ext}(\mathbb{C}) & \xrightarrow{F_1} & \mathrm{NExt}(\mathbb{C}) \\ {\scriptstyle \mathrm{Cod}}\downarrow & \nearrow & \downarrow{\scriptstyle \mathrm{Ker}} \\ \mathbb{C} & \underset{\pi_1^F}{\dashrightarrow} & \mathbb{X}. \end{array}$$

If $\mathbb{C}$ has enough projectives, and $\mathbb{X}$ is closed in $\mathbb{C}$ under regular quotients, then $\pi_1^F \cong \mathrm{L}_1 F$, the first simplicial left derived functor of F.

The fundamental group, I

Categorical Galois theory provides notions such as *central extensions* and *fundamental groups* relative to a suitable adjunction $F\colon \mathbb{C} \to \mathbb{X}$.

When $\mathbb{C}$ is semi-abelian and F is a regular epi–reflector which preserves pullbacks of regular epimorphisms along split epimorphisms, there is a derived reflection $F_1\colon \mathrm{Ext}(\mathbb{C}) \to \mathrm{NExt}(\mathbb{C})$.

- $\mathrm{NExt}(\mathbb{C})$ is the category of **normal extensions** relative to F.
- If $F = \mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$, then $F_1\colon \mathrm{Ext}(\mathrm{Gp}) \to \mathrm{CExt}(\mathrm{Gp})$ is the **centralisation functor** sending $f\colon X \to Y$ to $F_1(f)\colon \frac{X}{[X,\mathrm{Ker}(f)]} \to Y$.

The **fundamental group functor** of F is the right Kan extension π_1^F as in

$$\begin{array}{ccc} \mathrm{Ext}(\mathbb{C}) & \xrightarrow{F_1} & \mathrm{NExt}(\mathbb{C}) \\ {\scriptstyle \mathrm{Cod}}\downarrow & \nearrow & \downarrow{\scriptstyle \mathrm{Ker}} \\ \mathbb{C} & \underset{\pi_1^F}{\dashrightarrow} & \mathbb{X}. \end{array}$$

If $\mathbb{C}$ has enough projectives, and $\mathbb{X}$ is closed in $\mathbb{C}$ under regular quotients, then $\pi_1^F \cong \mathrm{L}_1 F$, the first simplicial left derived functor of F.

The fundamental group, I

Categorical Galois theory provides notions such as *central extensions* and *fundamental groups* relative to a suitable adjunction $F\colon \mathbb{C} \to \mathbb{X}$.

When $\mathbb{C}$ is semi-abelian and F is a regular epi–reflector which preserves pullbacks of regular epimorphisms along split epimorphisms, there is a derived reflection $F_1\colon \mathrm{Ext}(\mathbb{C}) \to \mathrm{NExt}(\mathbb{C})$.

- $\mathrm{NExt}(\mathbb{C})$ is the category of **normal extensions** relative to F.
- If $F = \mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$, then $F_1\colon \mathrm{Ext}(\mathrm{Gp}) \to \mathrm{CExt}(\mathrm{Gp})$ is the **centralisation functor** sending $f\colon X \to Y$ to $F_1(f)\colon \frac{X}{[X,\mathrm{Ker}(f)]} \to Y$.

The **fundamental group functor** of F is the right Kan extension π_1^F as in

$$
\begin{array}{ccc}
\mathrm{Ext}(\mathbb{C}) & \xrightarrow{F_1} & \mathrm{NExt}(\mathbb{C}) \\
{\scriptstyle \mathrm{Cod}}\downarrow & \nearrow\!\!\!\nearrow & \downarrow{\scriptstyle \mathrm{Ker}} \\
\mathbb{C} & \underset{\pi_1^F}{\dashrightarrow} & \mathbb{X}.
\end{array}
$$

If $\mathbb{C}$ has enough projectives, and $\mathbb{X}$ is closed in $\mathbb{C}$ under regular quotients, then $\pi_1^F \cong \mathrm{L}_1 F$, the first simplicial left derived functor of F.

The fundamental group, I [J Goedecke & TVdL, 2009]

Categorical Galois theory provides notions such as *central extensions* and *fundamental groups* relative to a suitable adjunction $F\colon \mathbb{C} \to \mathbb{X}$.

When $\mathbb{C}$ is semi-abelian and F is a regular epi–reflector which preserves pullbacks of regular epimorphisms along split epimorphisms, there is a derived reflection $F_1\colon \mathrm{Ext}(\mathbb{C}) \to \mathrm{NExt}(\mathbb{C})$.

- $\mathrm{NExt}(\mathbb{C})$ is the category of **normal extensions** relative to F.
- If $F = \mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$, then $F_1\colon \mathrm{Ext}(\mathrm{Gp}) \to \mathrm{CExt}(\mathrm{Gp})$ is the **centralisation functor** sending $f\colon X \to Y$ to $F_1(f)\colon \frac{X}{[X,\mathrm{Ker}(f)]} \to Y$.

The **fundamental group functor** of F is the right Kan extension π_1^F as in

$$\begin{array}{ccc} \mathrm{Ext}(\mathbb{C}) & \xrightarrow{F_1} & \mathrm{NExt}(\mathbb{C}) \\ {\scriptstyle \mathrm{Cod}}\downarrow & \nearrow\!\!\!\nearrow & \downarrow{\scriptstyle \mathrm{Ker}} \\ \mathbb{C} & \underset{\pi_1^F}{\dashrightarrow} & \mathbb{X}. \end{array}$$

If $\mathbb{C}$ has enough projectives, and $\mathbb{X}$ is closed in $\mathbb{C}$ under regular quotients, then $\pi_1^F \cong \mathrm{L}_1F$, the first simplicial left derived functor of F.

The fundamental group, I

Categorical Galois theory provides notions such as *central extensions* and *fundamental groups* relative to a suitable adjunction $F\colon \mathbb{C} \to \mathbb{X}$.

When $\mathbb{C}$ is semi-abelian and F is a regular epi–reflector which preserves pullbacks of regular epimorphisms along split epimorphisms, there is a derived reflection $F_1\colon \mathrm{Ext}(\mathbb{C}) \to \mathrm{NExt}(\mathbb{C})$.

- $\mathrm{NExt}(\mathbb{C})$ is the category of **normal extensions** relative to F.
- If $F = \mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$, then $F_1\colon \mathrm{Ext}(\mathrm{Gp}) \to \mathrm{CExt}(\mathrm{Gp})$ is the **centralisation functor** sending $f\colon X \to Y$ to $F_1(f)\colon \frac{X}{[X,\mathrm{Ker}(f)]} \to Y$.

The **fundamental group functor** of F is the right Kan extension π_1^F as in

$$\begin{array}{ccc} \mathrm{Ext}(\mathbb{C}) & \xrightarrow{F_1} & \mathrm{NExt}(\mathbb{C}) \\ {\scriptstyle \mathrm{Cod}}\downarrow & \Nearrow & \downarrow{\scriptstyle \mathrm{Ker}} \\ \mathbb{C} & \underset{\pi_1^F}{\dashrightarrow} & \mathbb{X}. \end{array}$$

If $\mathbb{C}$ has enough projectives, and $\mathbb{X}$ is closed in $\mathbb{C}$ under regular quotients, then $\pi_1^F \cong \mathrm{L}_1 F$, the first simplicial left derived functor of F.

The fundamental group, II

Taking the action of the centralisation functor as a definition of a commutator relative to the given adjunction, π_1^F looks as follows.

Let $p\colon P \to A$ be a projective cover with kernel K; apply F and F_1.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_1^F(A) & \rightarrowtail & \frac{[P,P]}{[P,K]} & \twoheadrightarrow & [A,A] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{K}{[P,K]} & \rightarrowtail & \frac{P}{[P,K]} & \xrightarrow{F_1(p)} & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{K}{K\cap[P,P]} & \rightarrowtail & \frac{P}{[P,P]} & \xrightarrow{F(p)} & \frac{A}{[A,A]} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The Hopf formula [T Everaert & TVdL, 2004] says $\pi_1^F(A) \cong \frac{K\cap[P,P]}{[P,K]}$.

Conclusion: $\pi_1^F(A)$ is the kernel of the morphism $\mathrm{Ker}(F_1(p)) \to F(P)$.

The fundamental group, II

Taking the action of the centralisation functor as a definition of a commutator relative to the given adjunction, π_1^F looks as follows.

Let $p\colon P \to A$ be a projective cover with kernel K; apply F and F_1.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_1^F(A) & \rightarrowtail & \frac{[P,P]}{[P,K]} & \twoheadrightarrow & [A,A] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{K}{[P,K]} & \rightarrowtail & \frac{P}{[P,K]} & \xrightarrow{F_1(p)} & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{K}{K\cap[P,P]} & \rightarrowtail & \frac{P}{[P,P]} & \xrightarrow{F(p)} & \frac{A}{[A,A]} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The Hopf formula [T Everaert & TVdL, 2004] says $\pi_1^F(A) \cong \frac{K\cap[P,P]}{[P,K]}$.

Conclusion: $\pi_1^F(A)$ is the kernel of the morphism $\mathrm{Ker}(F_1(p)) \to F(P)$.

The fundamental group, II

Taking the action of the centralisation functor as a definition of a commutator relative to the given adjunction, π_1^F looks as follows.

Let $p\colon P \to A$ be a projective cover with kernel K; apply F and F_1.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_1^F(A) & \rightarrowtail & \frac{[P,P]}{[P,K]} & \twoheadrightarrow & [A,A] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{K}{[P,K]} & \rightarrowtail & \frac{P}{[P,K]} & \xrightarrow{F_1(p)} & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{K}{K\cap[P,P]} & \rightarrowtail & \frac{P}{[P,P]} & \xrightarrow{F(p)} & \frac{A}{[A,A]} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

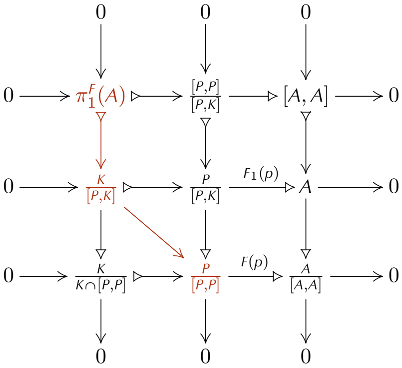
The Hopf formula [T Everaert & TVdL, 2004] says $\pi_1^F(A) \cong \frac{K\cap[P,P]}{[P,K]}$.

Conclusion: $\pi_1^F(A)$ is the kernel of the morphism $\mathrm{Ker}(F_1(p)) \to F(P)$.

The fundamental group, II

Taking the action of the centralisation functor as a definition of a commutator relative to the given adjunction, π_1^F looks as follows.

Let $p\colon P \to A$ be a projective cover with kernel K; apply F and F_1.



The Hopf formula [T Everaert & TVdL, 2004] says $\pi_1^F(A) \cong \frac{K\cap[P,P]}{[P,K]}$.

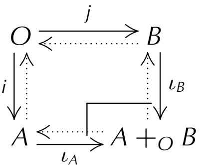
Conclusion: $\pi_1^F(A)$ is the kernel of the morphism $\mathrm{Ker}(F_1(p)) \to F(P)$.

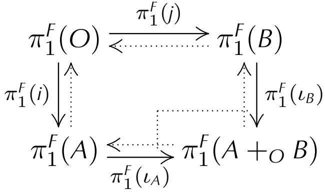
A Seifert–van Kampen theorem in non-abelian algebra

Let $F\colon \mathbb{C} \to \mathbb{X}$ be a functor where

- $\mathbb{C}$ is semi-abelian algebraically coherent with enough projectives;
- $\mathbb{X}$ is (almost) abelian;
- F is a regular epi–reflector which
  preserves pullbacks of regular epis along split epis

and consider in $\mathbb{C}$ a pushout of split monomorphisms as on the left.





Then the square on the right is a pushout square in $\mathbb{X}$.

Higher degrees?

If $O = 0$, then the result becomes $\pi_1^F(A+B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$.

This is what [M Barr & J Beck, 1969] call a *homology coproduct theorem*.

When A and B are groups, it is known that for all $n \geqslant 1$,

$$H_{n+1}(A+B,\mathbb{Z}) \cong H_{n+1}(A,\mathbb{Z}) \oplus H_{n+1}(B,\mathbb{Z}),$$

so is $F = \mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$, then $\pi_n^F(A+B) \cong \pi_n^F(A) \oplus \pi_n^F(B)$.

Is this a general fact? [KW Johnson, 1974]

$$\pi_2^G(C_2+C_2) = C_2 \oplus C_2 \oplus C_2 \ncong C_2 \oplus C_2 = \pi_2^G(C_2) \oplus \pi_2^G(C_2),$$

for $G = \mathrm{ab}\colon \mathrm{Nil}_2(\mathrm{Gp}) \to \mathrm{Ab}$ and C_2 the cyclic group of order two.
A general homology coproduct theorem applicable to G cannot exist!

This leaves open only two courses of investigation:

1. study case $n = 1$ in a setting wide enough to include $\mathrm{Nil}_2(\mathrm{Gp})$;
2. study the situation for general n, but in a more restricted context which includes Gp and rules out $\mathrm{Nil}_2(\mathrm{Gp})$. $\rightsquigarrow$ (LACC)?

Higher degrees?

If $O = 0$, then the result becomes $\pi_1^F(A+B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$.
This is what [M Barr & J Beck, 1969] call a *homology coproduct theorem*.

When A and B are groups, it is known that for all $n \geqslant 1$,

$$H_{n+1}(A+B,\mathbb{Z}) \cong H_{n+1}(A,\mathbb{Z}) \oplus H_{n+1}(B,\mathbb{Z}),$$

so is $F = \mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$, then $\pi_n^F(A+B) \cong \pi_n^F(A) \oplus \pi_n^F(B)$.

Is this a general fact? [KW Johnson, 1974]

$$\pi_2^G(C_2+C_2) = C_2 \oplus C_2 \oplus C_2 \ncong C_2 \oplus C_2 = \pi_2^G(C_2) \oplus \pi_2^G(C_2),$$

for $G = \mathrm{ab}\colon \mathrm{Nil}_2(\mathrm{Gp}) \to \mathrm{Ab}$ and C_2 the cyclic group of order two.
A general homology coproduct theorem applicable to G cannot exist!

This leaves open only two courses of investigation:

1. study case $n = 1$ in a setting wide enough to include $\mathrm{Nil}_2(\mathrm{Gp})$;
2. study the situation for general n, but in a more restricted context which includes Gp and rules out $\mathrm{Nil}_2(\mathrm{Gp})$. $\rightsquigarrow$ (LACC)?

Higher degrees?

If $O = 0$, then the result becomes $\pi_1^F(A+B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$.
This is what [M Barr & J Beck, 1969] call a *homology coproduct theorem*.

When A and B are groups, it is known that for all $n \geqslant 1$,

$$H_{n+1}(A+B,\mathbb{Z}) \cong H_{n+1}(A,\mathbb{Z}) \oplus H_{n+1}(B,\mathbb{Z}),$$

so is $F = \mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$, then $\pi_n^F(A+B) \cong \pi_n^F(A) \oplus \pi_n^F(B)$.

Is this a general fact? [KW Johnson, 1974]

$$\pi_2^G(C_2 + C_2) = C_2 \oplus C_2 \oplus C_2 \not\cong C_2 \oplus C_2 = \pi_2^G(C_2) \oplus \pi_2^G(C_2),$$

for $G = \mathrm{ab}\colon \mathrm{Nil}_2(\mathrm{Gp}) \to \mathrm{Ab}$ and C_2 the cyclic group of order two.
A general homology coproduct theorem applicable to G cannot exist!

This leaves open only two courses of investigation:

1. study case $n = 1$ in a setting wide enough to include $\mathrm{Nil}_2(\mathrm{Gp})$;
2. study the situation for general n, but in a more restricted context which includes Gp and rules out $\mathrm{Nil}_2(\mathrm{Gp})$. $\rightsquigarrow$ (LACC)?

Higher degrees?

If $O = 0$, then the result becomes $\pi_1^F(A+B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$.
This is what [M Barr & J Beck, 1969] call a *homology coproduct theorem*.

When A and B are groups, it is known that for all $n \geqslant 1$,

$$H_{n+1}(A+B,\mathbb{Z}) \cong H_{n+1}(A,\mathbb{Z}) \oplus H_{n+1}(B,\mathbb{Z}),$$

so is $F = \mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$, then $\pi_n^F(A+B) \cong \pi_n^F(A) \oplus \pi_n^F(B)$.

Is this a general fact? [KW Johnson, 1974]

$$\pi_2^G(C_2+C_2) = C_2 \oplus C_2 \oplus C_2 \ncong C_2 \oplus C_2 = \pi_2^G(C_2) \oplus \pi_2^G(C_2),$$

for $G = \mathrm{ab}\colon \mathrm{Nil}_2(\mathrm{Gp}) \to \mathrm{Ab}$ and C_2 the cyclic group of order two.
A general homology coproduct theorem applicable to G cannot exist!

This leaves open only two courses of investigation:

1. study case $n = 1$ in a setting wide enough to include $\mathrm{Nil}_2(\mathrm{Gp})$;
2. study the situation for general n, but in a more restricted context which includes Gp and rules out $\mathrm{Nil}_2(\mathrm{Gp})$. $\rightsquigarrow$ (LACC)?

Higher degrees?

If $O = 0$, then the result becomes $\pi_1^F(A+B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$.
This is what [M Barr & J Beck, 1969] call a *homology coproduct theorem*.

When A and B are groups, it is known that for all $n \geqslant 1$,

$$H_{n+1}(A+B,\mathbb{Z}) \cong H_{n+1}(A,\mathbb{Z}) \oplus H_{n+1}(B,\mathbb{Z}),$$

so is $F = \mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$, then $\pi_n^F(A+B) \cong \pi_n^F(A) \oplus \pi_n^F(B)$.

Is this a general fact? **No!** [KW Johnson, 1974]

$$\pi_2^G(C_2+C_2) = C_2 \oplus C_2 \oplus C_2 \not\cong C_2 \oplus C_2 = \pi_2^G(C_2) \oplus \pi_2^G(C_2),$$

for $G = \mathrm{ab}\colon \mathrm{Nil}_2(\mathrm{Gp}) \to \mathrm{Ab}$ and C_2 the cyclic group of order two.
A general homology coproduct theorem applicable to G cannot exist!

This leaves open only two courses of investigation:

1. study case $n = 1$ in a setting wide enough to include $\mathrm{Nil}_2(\mathrm{Gp})$;
2. study the situation for general n, but in a more restricted context which includes Gp and rules out $\mathrm{Nil}_2(\mathrm{Gp})$. $\rightsquigarrow$ (LACC)?

Higher degrees?

If $O = 0$, then the result becomes $\pi_1^F(A + B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$.
This is what [M Barr & J Beck, 1969] call a *homology coproduct theorem*.

When A and B are groups, it is known that for all $n \geqslant 1$,

$$H_{n+1}(A + B, \mathbb{Z}) \cong H_{n+1}(A, \mathbb{Z}) \oplus H_{n+1}(B, \mathbb{Z}),$$

so is $F = \mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$, then $\pi_n^F(A + B) \cong \pi_n^F(A) \oplus \pi_n^F(B)$.

Is this a general fact? **No!** [KW Johnson, 1974]

$$\pi_2^G(C_2 + C_2) = C_2 \oplus C_2 \oplus C_2 \not\cong C_2 \oplus C_2 = \pi_2^G(C_2) \oplus \pi_2^G(C_2),$$

for $G = \mathrm{ab}\colon \mathrm{Nil}_2(\mathrm{Gp}) \to \mathrm{Ab}$ and C_2 the cyclic group of order two.
A general homology coproduct theorem applicable to G cannot exist!

This leaves open only two courses of investigation:

1. study case $n = 1$ in a setting wide enough to include $\mathrm{Nil}_2(\mathrm{Gp})$;
2. study the situation for general n, but in a more restricted context which includes Gp and rules out $\mathrm{Nil}_2(\mathrm{Gp})$. $\rightsquigarrow$ (LACC)?

Higher degrees?

If $O = 0$, then the result becomes $\pi_1^F(A+B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$.
This is what [M Barr & J Beck, 1969] call a *homology coproduct theorem*.

When A and B are groups, it is known that for all $n \geqslant 1$,

$$H_{n+1}(A+B,\mathbb{Z}) \cong H_{n+1}(A,\mathbb{Z}) \oplus H_{n+1}(B,\mathbb{Z}),$$

so is $F = \mathrm{ab}\colon \mathsf{Gp} \to \mathsf{Ab}$, then $\pi_n^F(A+B) \cong \pi_n^F(A) \oplus \pi_n^F(B)$.

Is this a general fact? **No!** [KW Johnson, 1974]

$$\pi_2^G(C_2 + C_2) = C_2 \oplus C_2 \oplus C_2 \ncong C_2 \oplus C_2 = \pi_2^G(C_2) \oplus \pi_2^G(C_2),$$

for $G = \mathrm{ab}\colon \mathsf{Nil}_2(\mathsf{Gp}) \to \mathsf{Ab}$ and C_2 the cyclic group of order two.
A general homology coproduct theorem applicable to G cannot exist!

This leaves open only two courses of investigation:

1. study case $n = 1$ in a setting wide enough to include $\mathsf{Nil}_2(\mathsf{Gp})$;
2. study the situation for general n, but in a more restricted context which includes Gp and rules out $\mathsf{Nil}_2(\mathsf{Gp})$. $\rightsquigarrow$ (LACC)?

Higher degrees?

If $O = 0$, then the result becomes $\pi_1^F(A+B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$.
This is what [M Barr & J Beck, 1969] call a *homology coproduct theorem*.

When A and B are groups, it is known that for all $n \geqslant 1$,

$$H_{n+1}(A+B,\mathbb{Z}) \cong H_{n+1}(A,\mathbb{Z}) \oplus H_{n+1}(B,\mathbb{Z}),$$

so is $F = \mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$, then $\pi_n^F(A+B) \cong \pi_n^F(A) \oplus \pi_n^F(B)$.

Is this a general fact? **No!** [KW Johnson, 1974]

$$\pi_2^G(C_2 + C_2) = C_2 \oplus C_2 \oplus C_2 \not\cong C_2 \oplus C_2 = \pi_2^G(C_2) \oplus \pi_2^G(C_2),$$

for $G = \mathrm{ab}\colon \mathrm{Nil}_2(\mathrm{Gp}) \to \mathrm{Ab}$ and C_2 the cyclic group of order two.
A general homology coproduct theorem applicable to G cannot exist!

This leaves open only two courses of investigation:

1. study case $n = 1$ in a setting wide enough to include $\mathrm{Nil}_2(\mathrm{Gp})$;
2. study the situation for general n, but in a more restricted context which includes Gp and rules out $\mathrm{Nil}_2(\mathrm{Gp})$. $\rightsquigarrow$ (LACC)?

Higher degrees?

If $O = 0$, then the result becomes $\pi_1^F(A + B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$.
This is what [M Barr & J Beck, 1969] call a *homology coproduct theorem*.

When A and B are groups, it is known that for all $n \geqslant 1$,

$$H_{n+1}(A + B, \mathbb{Z}) \cong H_{n+1}(A, \mathbb{Z}) \oplus H_{n+1}(B, \mathbb{Z}),$$

so is $F = \mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$, then $\pi_n^F(A + B) \cong \pi_n^F(A) \oplus \pi_n^F(B)$.

Is this a general fact? **No!** [KW Johnson, 1974]

$$\pi_2^G(C_2 + C_2) = C_2 \oplus C_2 \oplus C_2 \ncong C_2 \oplus C_2 = \pi_2^G(C_2) \oplus \pi_2^G(C_2),$$

for $G = \mathrm{ab}\colon \mathrm{Nil}_2(\mathrm{Gp}) \to \mathrm{Ab}$ and C_2 the cyclic group of order two.
A general homology coproduct theorem applicable to G cannot exist!

This leaves open only two courses of investigation:

1. study case $n = 1$ in a setting wide enough to include $\mathrm{Nil}_2(\mathrm{Gp})$;
2. study the situation for general n, but in a more restricted context which includes Gp and rules out $\mathrm{Nil}_2(\mathrm{Gp})$. $\rightsquigarrow$ (LACC)?

Higher degrees?

If $O = 0$, then the result becomes $\pi_1^F(A + B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$.
This is what [M Barr & J Beck, 1969] call a *homology coproduct theorem*.

When A and B are groups, it is known that for all $n \geqslant 1$,

$$H_{n+1}(A + B, \mathbb{Z}) \cong H_{n+1}(A, \mathbb{Z}) \oplus H_{n+1}(B, \mathbb{Z}),$$

so is $F = \mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$, then $\pi_n^F(A + B) \cong \pi_n^F(A) \oplus \pi_n^F(B)$.

Is this a general fact? **No!** [KW Johnson, 1974]

$$\pi_2^G(C_2 + C_2) = C_2 \oplus C_2 \oplus C_2 \ncong C_2 \oplus C_2 = \pi_2^G(C_2) \oplus \pi_2^G(C_2),$$

for $G = \mathrm{ab}\colon \mathrm{Nil}_2(\mathrm{Gp}) \to \mathrm{Ab}$ and C_2 the cyclic group of order two.
A general homology coproduct theorem applicable to G cannot exist!

This leaves open only two courses of investigation:

1. study case $n = 1$ in a setting wide enough to include $\mathrm{Nil}_2(\mathrm{Gp})$;
2. study the situation for general n, but in a more restricted context which includes Gp and rules out $\mathrm{Nil}_2(\mathrm{Gp})$. $\rightsquigarrow$ (LACC)?

Higher degrees?

If $O = 0$, then the result becomes $\pi_1^F(A + B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$.
This is what [M Barr & J Beck, 1969] call a *homology coproduct theorem*.

When A and B are groups, it is known that for all $n \geqslant 1$,

$$H_{n+1}(A + B, \mathbb{Z}) \cong H_{n+1}(A, \mathbb{Z}) \oplus H_{n+1}(B, \mathbb{Z}),$$

so is $F = \mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$, then $\pi_n^F(A + B) \cong \pi_n^F(A) \oplus \pi_n^F(B)$.

Is this a general fact? **No!** [KW Johnson, 1974]

$$\pi_2^G(C_2 + C_2) = C_2 \oplus C_2 \oplus C_2 \not\cong C_2 \oplus C_2 = \pi_2^G(C_2) \oplus \pi_2^G(C_2),$$

for $G = \mathrm{ab}\colon \mathrm{Nil}_2(\mathrm{Gp}) \to \mathrm{Ab}$ and C_2 the cyclic group of order two.
A general homology coproduct theorem applicable to G cannot exist!

This leaves open only two courses of investigation:

1. study case $n = 1$ in a setting wide enough to include $\mathrm{Nil}_2(\mathrm{Gp})$;
2. study the situation for general n, but in a more restricted context which includes Gp and rules out $\mathrm{Nil}_2(\mathrm{Gp})$. $\rightsquigarrow$ (LACC)?

The semi-abelian context

Category is	Noteworthy aspects	Examples
abelian	classical homological algebra additive structure on homsets subobjects are normal	Ab, Mod_R, $\mathrm{Sh}_{\mathscr{T}}(\mathrm{Ab})$
semi-abelian + (LACC)	internal (co)homology	Gp, XMod, Lie_K, $\mathrm{HopfAlg}_{K,\mathrm{coc}}$
semi-abelian + algebraically coherent	Seifert–van Kampen universal central extensions three subobjects lemma	$\mathrm{Nil}_n(\mathrm{Gp})$, $\mathrm{Gp}_{\mathrm{tf}}$, $\mathrm{Cat}^n(\mathrm{Gp})$ all *categories of interest*: Rng, Alg_K, Leib_K
semi-abelian + (SH)	internal crossed modules action on abelian object is Beck module cohomology via higher centrality	NAAlg_K, Jordan algebras, Heyting semilattices
semi-abelian	basic homological algebra internal actions	Loop, DiGp all *varieties of* Ω*-groups*

- **semi-abelian** = pointed + Barr exact + protomodular + finite sums
- Internal actions correspond to split extensions: if B acts on X via ξ,

$$0 \longrightarrow X \rightarrowtail X \rtimes_\xi B \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0.$$

  Thus an action corresponds to a **point**: a couple (f, s) with $fs = 1_B$.
  Think of a point as a "non-abelian module".
- commutators measure non-abelianness $\rightsquigarrow$ commutator conditions

Algebraic coherence

[AS Cigoli, JRA Gray & TVdL, 2015]

Internal actions correspond to split extensions: for B acting on X via ξ,

$$0 \longrightarrow X \rightarrowtail X \rtimes_{\xi} B \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0.$$

Thus an action corresponds to a **point**: a couple (f, s) with $fs = 1_B$.

When the underlying object of a sum of B-actions $(X, \xi) + (Y, \upsilon)$ is the sum $X + Y$, this means that the category $\mathbb{C}$ is **(LACC)**.

When the canonical comparison $X + Y \to K((X, \xi) + (Y, \upsilon))$ is merely a regular epimorphism (instead of an isomorphism), then the category $\mathbb{C}$ is **algebraically coherent**.

Thus, algebraic coherence means that the change-of-base functors in the fibration of points preserve jointly strongly epimorphic pairs: **if** (X, ξ) **and** (Y, υ) **cover** (Z, ζ) **in** $\mathrm{Act}_B(\mathbb{C})$**, then** X **and** Y **cover** Z **in** $\mathbb{C}$.

This technical condition has strong categorical-algebraic consequences, while (in contrast with (LACC)) many semi-abelian categories satisfy it.

Algebraic coherence

[AS Cigoli, JRA Gray & TVdL, 2015]

Internal actions correspond to split extensions: for B acting on X via ξ,

$$0 \longrightarrow X \rightarrowtriangle X \rtimes_{\xi} B \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0.$$

Thus an action corresponds to a **point**: a couple (f, s) with $fs = 1_B$.

When the underlying object of a sum of B-actions $(X, \xi) + (Y, \upsilon)$ is the sum $X + Y$, this means that the category $\mathbb{C}$ is **(LACC)**.

When the canonical comparison $X + Y \to K((X, \xi) + (Y, \upsilon))$ is merely a regular epimorphism (instead of an isomorphism), then the category $\mathbb{C}$ is **algebraically coherent**.

Thus, algebraic coherence means that the change-of-base functors in the fibration of points preserve jointly strongly epimorphic pairs: **if** (X, ξ) **and** (Y, υ) **cover** (Z, ζ) **in** $\mathrm{Act}_B(\mathbb{C})$**, then** X **and** Y **cover** Z **in** $\mathbb{C}$.

This technical condition has strong categorical-algebraic consequences, while (in contrast with (LACC)) many semi-abelian categories satisfy it.

Algebraic coherence

[AS Cigoli, JRA Gray & TVdL, 2015]

Internal actions correspond to split extensions: for B acting on X via ξ,

$$0 \longrightarrow X \rightarrowtail X \rtimes_{\xi} B \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0.$$

Thus an action corresponds to a **point**: a couple (f, s) with $fs = 1_B$.

When the underlying object of a sum of B-actions $(X, \xi) + (Y, \upsilon)$ is the sum $X + Y$, this means that the category $\mathbb{C}$ is **(LACC)**.

When the canonical comparison $X + Y \to K((X, \xi) + (Y, \upsilon))$ is merely a regular epimorphism (instead of an isomorphism), then the category $\mathbb{C}$ is **algebraically coherent**.

Thus, algebraic coherence means that the change-of-base functors in the fibration of points preserve jointly strongly epimorphic pairs: **if** (X, ξ) **and** (Y, υ) **cover** (Z, ζ) **in** $\mathrm{Act}_B(\mathbb{C})$**, then** X **and** Y **cover** Z **in** $\mathbb{C}$.

This technical condition has strong categorical-algebraic consequences, while (in contrast with (LACC)) many semi-abelian categories satisfy it.

Algebraic coherence

[AS Cigoli, JRA Gray & TVdL, 2015]

Internal actions correspond to split extensions: for B acting on X via ξ,

$$0 \longrightarrow X \rightarrowtriangle X \rtimes_{\xi} B \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0.$$

Thus an action corresponds to a **point**: a couple (f, s) with $fs = 1_B$.

When the underlying object of a sum of B-actions $(X, \xi) + (Y, \upsilon)$ is the sum $X + Y$, this means that the category $\mathbb{C}$ is **(LACC)**.

When the canonical comparison $X + Y \to K((X, \xi) + (Y, \upsilon))$ is merely a regular epimorphism (instead of an isomorphism), then the category $\mathbb{C}$ is **algebraically coherent**.

Thus, algebraic coherence means that the change-of-base functors in the fibration of points preserve jointly strongly epimorphic pairs: **if** (X, ξ) **and** (Y, υ) **cover** (Z, ζ) **in** $\mathrm{Act}_B(\mathbb{C})$**, then** X **and** Y **cover** Z **in** $\mathbb{C}$.

This technical condition has strong categorical-algebraic consequences, while (in contrast with (LACC)) many semi-abelian categories satisfy it.

Algebraic coherence

[AS Cigoli, JRA Gray & TVdL, 2015]

Internal actions correspond to split extensions: for B acting on X via ξ,

$$0 \longrightarrow X \rightarrowtail X \rtimes_{\xi} B \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0.$$

Thus an action corresponds to a **point**: a couple (f, s) with $fs = 1_B$.

When the underlying object of a sum of B-actions $(X, \xi) + (Y, \upsilon)$ is the sum $X + Y$, this means that the category $\mathbb{C}$ is **(LACC)**.

When the canonical comparison $X + Y \to K((X, \xi) + (Y, \upsilon))$ is merely a regular epimorphism (instead of an isomorphism), then the category $\mathbb{C}$ is **algebraically coherent**.

Thus, algebraic coherence means that the change-of-base functors in the fibration of points preserve jointly strongly epimorphic pairs: **if (X, ξ) and (Y, υ) cover (Z, ζ) in $\mathrm{Act}_B(\mathbb{C})$, then X and Y cover Z in $\mathbb{C}$.**

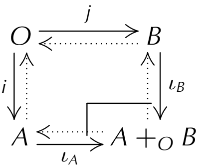
This technical condition has strong categorical-algebraic consequences, while (in contrast with (LACC)) many semi-abelian categories satisfy it.

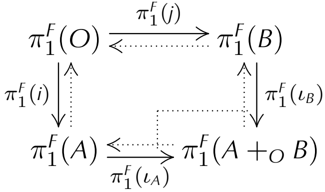
A Seifert–van Kampen theorem in non-abelian algebra

Let $F\colon \mathbb{C} \to \mathbb{X}$ be a functor where

- $\mathbb{C}$ is semi-abelian algebraically coherent with enough projectives;
- $\mathbb{X}$ is (almost) abelian;
- F is a regular epi–reflector which
  preserves pullbacks of regular epis along split epis

and consider in $\mathbb{C}$ a pushout of split monomorphisms as on the left.





Then the square on the right is a pushout square in $\mathbb{X}$.

A glance at the proof of $\pi_1^F(A+B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$

We take projective presentations:

$$\begin{array}{ccccc} P & \rightleftarrows & P+Q & \rightleftarrows & Q \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p+q} & & \downarrow{\scriptstyle q} \\ A & \rightleftarrows & A+B & \rightleftarrows & B \end{array}$$

Since $\pi_1^F(A)$ is the kernel of $\mathrm{Ker}(F_1(p)) \to F(P)$, we consider

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \pi_1^F(A) & \rightleftarrows & \pi_1^F(A+B) & \rightleftarrows & \pi_1^F(B) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathrm{Ker}(F_1(p)) & \rightleftarrows & \mathrm{Ker}(F_1(p+q)) & \rightleftarrows & \mathrm{Ker}(F_1(q)) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F(P) & \rightleftarrows & F(P+Q) & \rightleftarrows & F(Q) & \longrightarrow & 0 \end{array}$$

We want a biproduct in the top row. The bottom row is a biproduct.
We only need to prove that the second row is a biproduct:
this follows from algebraic coherence.

A glance at the proof of $\pi_1^F(A+B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$

We take projective presentations:

$$
\begin{array}{ccccc}
P & \rightleftarrows & P+Q & \rightleftarrows & Q \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p+q} & & \downarrow{\scriptstyle q} \\
A & \rightleftarrows & A+B & \rightleftarrows & B
\end{array}
$$

Since $\pi_1^F(A)$ is the kernel of $\mathrm{Ker}(F_1(p)) \to F(P)$, we consider

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \pi_1^F(A) & \rightleftarrows & \pi_1^F(A+B) & \rightleftarrows & \pi_1^F(B) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{Ker}(F_1(p)) & \rightleftarrows & \mathrm{Ker}(F_1(p+q)) & \rightleftarrows & \mathrm{Ker}(F_1(q)) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(P) & \rightleftarrows & F(P+Q) & \rightleftarrows & F(Q) & \longrightarrow & 0
\end{array}
$$

We want a biproduct in the top row. The bottom row is a biproduct.
We only need to prove that the second row is a biproduct:
this follows from algebraic coherence.

A glance at the proof of $\pi_1^F(A+B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$

We take projective presentations:

$$\begin{array}{ccccc} P & \leftrightarrows & P+Q & \rightleftarrows & Q \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p+q} & & \downarrow{\scriptstyle q} \\ A & \leftrightarrows & A+B & \rightleftarrows & B \end{array}$$

Since $\pi_1^F(A)$ is the kernel of $\mathrm{Ker}(F_1(p)) \to F(P)$, we consider

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \pi_1^F(A) & \rightleftarrows & \pi_1^F(A+B) & \rightleftarrows & \pi_1^F(B) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathrm{Ker}(F_1(p)) & \rightleftarrows & \mathrm{Ker}(F_1(p+q)) & \rightleftarrows & \mathrm{Ker}(F_1(q)) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F(P) & \rightleftarrows & F(P+Q) & \rightleftarrows & F(Q) & \longrightarrow & 0 \end{array}$$

We want a biproduct in the top row. The bottom row is a biproduct.
We only need to prove that the second row is a biproduct:
this follows from algebraic coherence.

A glance at the proof of $\pi_1^F(A+B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$

We take projective presentations:

$$\begin{array}{ccccc} P & \leftrightarrows & P+Q & \rightleftarrows & Q \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p+q} & & \downarrow{\scriptstyle q} \\ A & \leftrightarrows & A+B & \rightleftarrows & B \end{array}$$

Since $\pi_1^F(A)$ is the kernel of $\mathrm{Ker}(F_1(p)) \to F(P)$, we consider

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \pi_1^F(A) & \rightleftarrows & \pi_1^F(A+B) & \rightleftarrows & \pi_1^F(B) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathrm{Ker}(F_1(p)) & \rightleftarrows & \mathrm{Ker}(F_1(p+q)) & \rightleftarrows & \mathrm{Ker}(F_1(q)) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F(P) & \rightleftarrows & F(P+Q) & \rightleftarrows & F(Q) & \longrightarrow & 0 \end{array}$$

We want a biproduct in the top row. The bottom row is a biproduct.

We only need to prove that the second row is a biproduct:
this follows from algebraic coherence.

A glance at the proof of $\pi_1^F(A+B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$

We take projective presentations:

$$\begin{array}{ccccc}
P & \leftrightarrows & P+Q & \rightleftarrows & Q \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p+q} & & \downarrow{\scriptstyle q} \\
A & \leftrightarrows & A+B & \rightleftarrows & B
\end{array}$$

Since $\pi_1^F(A)$ is the kernel of $\mathrm{Ker}(F_1(p)) \to F(P)$, we consider

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \pi_1^F(A) & \rightleftarrows & \pi_1^F(A+B) & \rightleftarrows & \pi_1^F(B) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{Ker}(F_1(p)) & \rightleftarrows & \mathrm{Ker}(F_1(p+q)) & \rightleftarrows & \mathrm{Ker}(F_1(q)) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(P) & \rightleftarrows & F(P+Q) & \rightleftarrows & F(Q) & \longrightarrow & 0
\end{array}$$

We want a biproduct in the top row. The bottom row is a biproduct.
We only need to prove that the second row is a biproduct:
this follows from algebraic coherence.

A glance at the proof of $\pi_1^F(A+B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$

We take projective presentations:

$$\begin{array}{ccccc} P & \rightleftarrows & P+Q & \rightleftarrows & Q \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p+q} & & \downarrow{\scriptstyle q} \\ A & \rightleftarrows & A+B & \rightleftarrows & B \end{array}$$

Since $\pi_1^F(A)$ is the kernel of $\mathrm{Ker}(F_1(p)) \to F(P)$, we consider

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \pi_1^F(A) & \rightleftarrows & \pi_1^F(A+B) & \rightleftarrows & \pi_1^F(B) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathrm{Ker}(F_1(p)) & \rightleftarrows & \mathrm{Ker}(F_1(p+q)) & \rightleftarrows & \mathrm{Ker}(F_1(q)) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F(P) & \rightleftarrows & F(P+Q) & \rightleftarrows & F(Q) & \longrightarrow & 0 \end{array}$$

We want a biproduct in the top row. The bottom row is a biproduct.
We only need to prove that the second row is a biproduct:
this follows from algebraic coherence.

A glance at the proof of $\pi_1^F(A+B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$

The diagram

$$\mathrm{Ker}(F_1(p)) \rightleftarrows \mathrm{Ker}(F_1(p+q)) \rightleftarrows \mathrm{Ker}(F_1(q))$$

is a biproduct by the next result.

Proposition

In an algebraically coherent semi-abelian category, consider:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K' & \rhd\!\!\longrightarrow & Z & \rightleftarrows & T & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle t} & & \\
0 & \longrightarrow & K & \overset{k}{\rhd\!\!\longrightarrow} & X & \underset{s}{\overset{f}{\rightleftarrows}} & Y & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \kappa'} & & \uparrow{\scriptstyle z'} & & \uparrow{\scriptstyle t'} & & \\
0 & \longrightarrow & K'' & \rhd\!\!\longrightarrow & Z' & \rightleftarrows & T' & \longrightarrow & 0
\end{array}$$

If f is a normal extension, κ and κ' are split monomorphisms and (z, z') is jointly strongly epimorphic, then (κ, κ') is jointly strongly epimorphic.

- This argument doesn't this extend to higher degrees.

A glance at the proof of $\pi_1^F(A+B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$

The diagram

$$\mathrm{Ker}(F_1(p)) \rightleftarrows \mathrm{Ker}(F_1(p+q)) \rightleftarrows \mathrm{Ker}(F_1(q))$$

is a biproduct by the next result.

Proposition

In an algebraically coherent semi-abelian category, consider:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K' & \rightarrowtail & Z & \rightleftarrows & T & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle t} & & \\
0 & \longrightarrow & K & \overset{k}{\rightarrowtail} & X & \underset{s}{\overset{f}{\rightleftarrows}} & Y & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \kappa'} & & \uparrow{\scriptstyle z'} & & \uparrow{\scriptstyle t'} & & \\
0 & \longrightarrow & K'' & \rightarrowtail & Z' & \rightleftarrows & T' & \longrightarrow & 0
\end{array}$$

If f is a normal extension, κ and κ' are split monomorphisms and (z, z') is jointly strongly epimorphic, then (κ, κ') is jointly strongly epimorphic.

- This argument doesn't this extend to higher degrees.

A glance at the proof of $\pi_1^F(A+B) \cong \pi_1^F(A) \oplus \pi_1^F(B)$

The diagram

$$\mathsf{Ker}(F_1(p)) \rightleftarrows \mathsf{Ker}(F_1(p+q)) \rightleftarrows \mathsf{Ker}(F_1(q))$$

is a biproduct by the next result.

Proposition

In an algebraically coherent semi-abelian category, consider:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K' & \rightarrowtail & Z & \rightleftarrows & T & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle t} & & \\
0 & \longrightarrow & K & \overset{k}{\rightarrowtail} & X & \underset{s}{\overset{f}{\rightleftarrows}} & Y & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \kappa'} & & \uparrow{\scriptstyle z'} & & \uparrow{\scriptstyle t'} & & \\
0 & \longrightarrow & K'' & \rightarrowtail & Z' & \rightleftarrows & T' & \longrightarrow & 0
\end{array}$$

If f is a normal extension, κ and κ' are split monomorphisms and (z, z') is jointly strongly epimorphic, then (κ, κ') is jointly strongly epimorphic.

- This argument doesn't this extend to higher degrees.

Conclusion

Theorem

For F any reflector from a semi-abelian algebraically coherent variety to an abelian subvariety, its first left derived functor $\mathsf{L}_1 F$ preserves pushouts of split monomorphisms.

- Main point for me: role of categorical-algebraic conditions such as *algebraic coherence*.

  This ingredient is missing in, for instance, [M Barr & J Beck, 1969].
- Concrete applications?
- What is the right context for a higher-order result?
- Beyond split monomorphisms?

Conclusion

Theorem

For F any reflector from a semi-abelian algebraically coherent variety to an abelian subvariety, its first left derived functor $\mathsf{L}_1 F$ preserves pushouts of split monomorphisms.

- Main point for me: role of categorical-algebraic conditions such as *algebraic coherence*.
  This ingredient is missing in, for instance, [M Barr & J Beck, 1969].
- Concrete applications?
- What is the right context for a higher-order result?
- Beyond split monomorphisms?

Conclusion

Theorem

For F any reflector from a semi-abelian algebraically coherent variety to an abelian subvariety, its first left derived functor $\mathsf{L}_1 F$ preserves pushouts of split monomorphisms.

- Main point for me: role of categorical-algebraic conditions such as *algebraic coherence*.

  This ingredient is missing in, for instance, [M Barr & J Beck, 1969].
- Concrete applications?
- What is the right context for a higher-order result?
- Beyond split monomorphisms?

Conclusion

Theorem

For F any reflector from a semi-abelian algebraically coherent variety to an abelian subvariety, its first left derived functor $\mathrm{L}_1 F$ preserves pushouts of split monomorphisms.

- Main point for me: role of categorical-algebraic conditions such as *algebraic coherence*.

  This ingredient is missing in, for instance, [M Barr & J Beck, 1969].
- Concrete applications?
- What is the right context for a higher-order result?
- Beyond split monomorphisms?

Conclusion

Theorem

For F any reflector from a semi-abelian algebraically coherent variety to an abelian subvariety, its first left derived functor $\mathsf{L}_1 F$ preserves pushouts of split monomorphisms.

- Main point for me: role of categorical-algebraic conditions such as *algebraic coherence*.
  This ingredient is missing in, for instance, [M Barr & J Beck, 1969].
- Concrete applications?
- What is the right context for a higher-order result?
- Beyond split monomorphisms?

Thank you!