AUs

Sketches for Arithmetic Universes: Infinitary disjunctions in finitary form

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Toposes as spaces

Space = geometric theory $\mathbb{T}$

Map = geometric morphism $\mathcal{S}[\mathbb{T}_1] \to \mathcal{S}[\mathbb{T}_2]$

= model of $\mathbb{T}_2$ in $\mathcal{S}[\mathbb{T}_1]$

Let M be model of $\mathbb{T}_1$ Then $f(M) = \dots\dots$ is model of $\mathbb{T}_2$

using geometric maths

Formal system? - Difficult

Infinite disjunctions:

infinities extrinsic to logic

- supplied by base topos $\mathcal{S}$

IDEA

AUs have some intrinsic infinities

e.g. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$

Enough for point-free $\mathbb{R}$

Arithmetic Universes

Joyal
Maietti
[Wraith]

Pretopos
+ parametrized list objects

$\Rightarrow$ finite colimits

e.g. $\mathbb{N} = \mathrm{List}(1)$

Cartesian theory of AUs

$\therefore$ presentation $\mathbb{T} \longmapsto AU\langle\mathbb{T}\rangle$

cf. $\mathbb{T} \longmapsto \mathcal{S}[\mathbb{T}]$

~~Toposes~~ AUs as spaces

arithmetic

Space = ~~geometric~~ theory $\mathbb{T}$

AU functor $\qquad \mathrm{AU}\langle\mathbb{T}_2\rangle \longleftarrow \mathrm{AU}\langle\mathbb{T}_2\rangle$

Map = ~~geometric morphism~~ $\mathcal{S}[\mathbb{T}_1] \to \mathcal{S}[\mathbb{T}_2]$

= model of $\mathbb{T}_2$ in $\cancel{\mathcal{S}[\mathbb{T}_1]}$ $\mathrm{AU}\langle\mathbb{T}_1\rangle$

$$\boxed{\begin{array}{l}\text{Let } M \text{ be model of } \mathbb{T}_1 \\ \text{Then } f(M) = \dots\dots \\ \quad \text{is model of } \mathbb{T}_2\end{array}}$$

arithmetic

using ~~geometric~~ maths

Sketches

$\mathbb{T}$

$$
\begin{tikzcd}
\mathcal{U}^{pb} \arrow[d, shift right] \arrow[d, shift left] & \mathcal{U}^{\ell} \arrow[l] \arrow[r] \arrow[d, shift right, "cons"'] \arrow[d, shift left, "\varepsilon"] & \mathcal{U}^{1} \arrow[d] \\
G^2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G^1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & G^0 \arrow[l, "s"'] \\
\mathcal{U}^{2o} \arrow[u, shift right] \arrow[u, shift left] & & \mathcal{U}^{0} \arrow[u]
\end{tikzcd}
$$

Sketch homomorphisms

$$\Pi_1 \longleftarrow \Pi_2$$

gives

$$AU\langle \Pi_1 \rangle \longleftarrow AU\langle \Pi_2 \rangle$$

Strictness

Models — non-strict — semantics

Models — strict — syntax, universal algebra

unuseful object equations

e.g. pb P $\cong$ List(A)

Extensions

$\Pi \subset \Pi + \delta\Pi$

Universals
- only for fresh structure $\delta\Pi$

Extension
= finite sequence of simple extensions

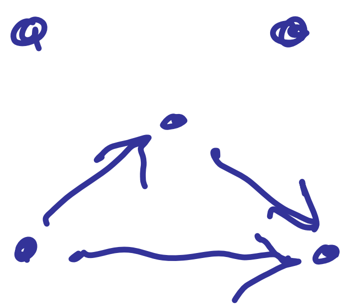
Context = extension of $\emptyset$

Simple extensions

$\mathbb{T} \subset \mathbb{T} + \delta\mathbb{T}$

Data

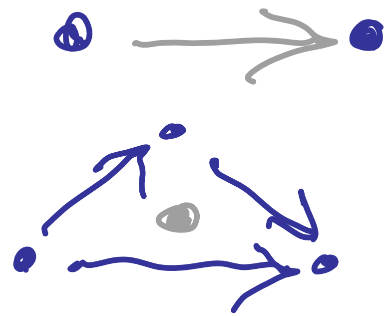
nothing



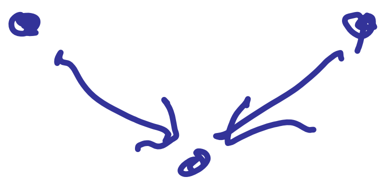
Delta

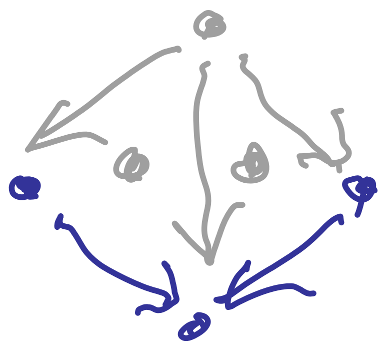
id

primitive node



Universals - e.g. pullback





Coherence

Theorem For a context Π, every non-strict model has an isomorphism with a strict model that is unique, subject to agreement on primitive nodes.

Reindexing

Substitution

extensions along sketch homs

Hom takes extension data to extension data

∴ transports extensions

$$\begin{tikzcd} {} & {} \arrow[l] \\ {} \arrow[u, "f(c)"] & {} \arrow[l, "f"] \arrow[u, "c"'] \end{tikzcd}$$

Will give strict pullbacks of extension maps

Sketch homomorphisms

Not general enough

Want:

Π_1' – stuff derived from Π_1

Con$_{hom}$

$$\begin{tikzcd}
 & \Pi_1' & \\
\Pi_1 & & \Pi_2 \arrow[ll] \arrow[ul]
\end{tikzcd}$$

gives

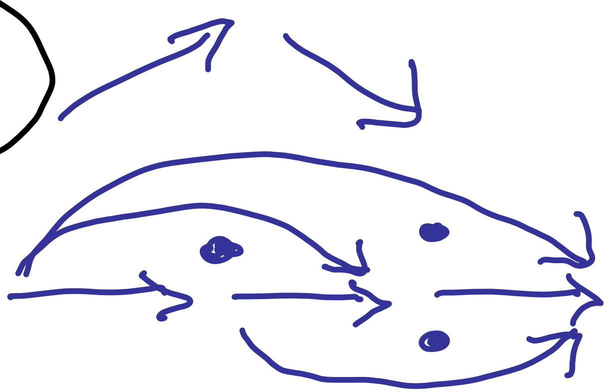
$$\begin{tikzcd}
 & AU\langle\Pi_1'\rangle & \\
AU\langle\Pi_1\rangle \arrow[ur, equal] & & AU\langle\Pi_2\rangle \arrow[ll] \arrow[ul]
\end{tikzcd}$$

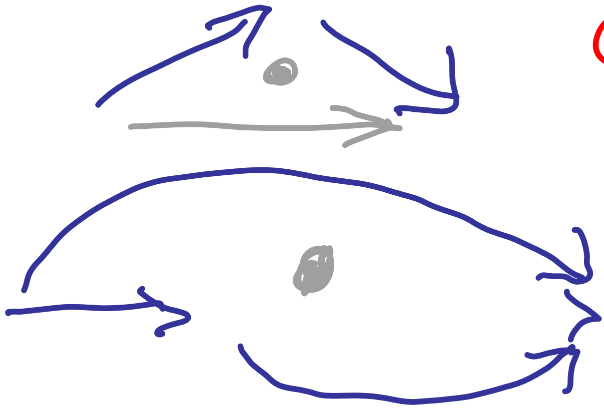
Equivalence extensions –
adjoin derived stuff

Equivalence extensions $\Pi \subseteq \Pi + \delta\Pi$

Data | **Delta**

e.g.





composition

associativity

as before

p.b. universals

p.b. fillins

p.b. fillin uniqueness

+ adjoining inverses for balance, stability, exactness

Object equalities

$X \Rightarrow Y$

Either $X \xrightarrow{id} X$

or - same construction done twice on equal data

Identity in any strict model

Context category

$$\mathrm{Con}^{op}_{hom} \to \mathrm{Con}$$

- dualize homomorphisms to maps
- invert equivalence extensions
- object equalities become identities

Map $\Pi_1 \to \Pi_2$:

$$\Pi_1 \hookrightarrow \Pi_1' \xleftarrow{\text{hom}} \Pi_2$$

Con is 2-category

- has finite pie limits
- strict pb of extension maps
- full & faithful embedding in AU_s^{op}

Aims

- Single AU-proof
  $\Rightarrow$ base-independent proof for Grothendieck toposes
- Extensions as bundles, fibrewise topology
- Constructive real analysis