Frames as topological algebras

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joint work with Giulia Frosoni and Alessio Santamaria

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Overview of the talk

- The cartesian closed category of the equilogical spaces

- The monad Σ^2 on the Sierpinski space Σ

- The comparision between the algebras for Σ^2 and frames

Algebraic Lattices

In a complete poset $(P, \leq)$, an element $c \in P$ is *compact* if, for every directed subset $X \subseteq P$,

$$\text{if } c \leq \bigvee^{\uparrow} X, \text{ then there is } x \in X \text{ such that } c \leq x.$$

A complete lattice $\mathscr{A} = (|\mathscr{A}|, \leq_{\mathscr{A}})$ is *algebraic* if every element is the directed join of the compact elements below it.

The compact elements are closed under finite joins.

∴ An algebraic lattice is isomorphic to the order of the ideals of its compact elements.

A function between algebraic lattices is *Scott-continuous* if it preserves joins of directed subsets.

The category $\mathcal{AlgLatt}$ of algebraic lattices and Scott-continuous functions is cartesian closed.

Equilogical spaces

as partial equivalence relations on algebraic lattices

equilogical space: $\mathbf{E} = (\mathscr{A}_\mathbf{E}, \smile_\mathbf{E})$ where

- $\mathscr{A}_\mathbf{E} = (|\mathscr{A}_\mathbf{E}|, \leq_{\mathscr{A}_\mathbf{E}})$ is an algebraic lattice
- $\smile_\mathbf{E}$ is a symmetric and transitive relation on $|\mathscr{A}_\mathbf{E}|$

map of equilogical spaces: $\mathbf{E} \xrightarrow{[f]_\equiv} \mathbf{D}$ is an equivalence class

of Scott-continuous functions $\mathscr{A}_\mathbf{E} \xrightarrow{f} \mathscr{A}_\mathbf{D}$ such that

$$\text{whenever } a \smile_\mathbf{E} b \text{ it is also } f(a) \smile_\mathbf{D} f(b).$$

Two such f and f' are equivalent $f \equiv f'$ when

$$\text{for every } a \smile_\mathbf{D} a, \text{ it is } f(a) \smile_\mathbf{D} f'(a).$$

$\mathcal{E}qu$: the category of equilogical spaces and maps between them.

Bauer, A., Birkedal, L., Scott, D.S.
Equilogical spaces. Theoret. Comput. Sci. 315 (2004)

$\mathcal{E}qu$ is a cartesian closed category

$\mathbf{T} = (\{*\}, =)$

$\mathbf{E} \times \mathbf{D} = (\mathscr{A}_{\mathbf{E}} \times \mathscr{A}_{\mathbf{D}}, \smile_{\mathbf{E}\times\mathbf{D}})$
where $\smile_{\mathbf{E}\times\mathbf{D}}$ is defined componentwise.

$\mathbf{D}^{\mathbf{E}} = ({\mathscr{A}_{\mathbf{D}}}^{\mathscr{A}_{\mathbf{E}}}, \smile_{\mathbf{D}^{\mathbf{E}}})$

where, for $\mathscr{A}_{\mathbf{E}} \overset{f}{\underset{g}{\rightrightarrows}} \mathscr{A}_{\mathbf{D}}$, $f \smile_{\mathbf{D}^{\mathbf{E}}} g$ is defined as follows:

for every $a, b \in |\mathscr{A}_{\mathbf{E}}|$, if $a \smile_{\mathbf{E}} b$ then $f(a) \smile_{\mathbf{D}} g(b)$.

Equilogical spaces and sets

The "global section" functor $\hom_{\mathcal{E}qu}(\mathbf{T}, -)$ is isomorphic to

$$
\begin{array}{ccc}
\mathcal{E}qu & \xrightarrow{\quad\Gamma\quad} & \mathcal{S}et \\
\mathbf{E} & \longmapsto & \{a \in |\mathscr{A}_{\mathbf{E}}| \mid a \smile_{\mathbf{E}} a\} \big/ \smile_{\mathbf{E}} \\
\downarrow \; [f]_{\equiv} & \longmapsto & \bar{f} \; \downarrow \\
\mathbf{D} & \longmapsto & \{b \in |\mathscr{A}_{\mathbf{D}}| \mid b \smile_{\mathbf{D}} b\} \big/ \smile_{\mathbf{D}}
\end{array}
$$

Since $\mathbf{T}$ is projective in $\mathcal{E}qu$, Γ preserves quotients.

Since algebraic lattices are injective in $\mathcal{T}op_0$, every continuous map

$$S \xrightarrow{\quad h \quad} S'$$

Equilogical spaces and T_0-spaces

Since algebraic lattices are injective in $\mathcal{T}op_0$, every continuous map can be extended continuously along the embeddings

$$\begin{array}{ccccc} x & S & \xrightarrow{\quad h \quad} & S' & x' \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ \mathcal{U}_x & \mathscr{P}(\tau_S) & \xrightarrow{\quad h' \quad} & \mathscr{P}(\tau_{S'}) & \mathcal{U}_{x'} \end{array}$$

If we write $=_S$ for the restriction of the identity relation of $\mathscr{P}(\tau_S)$ to S, the diagram above gives a map of equilogical spaces

$$(\mathscr{P}(\tau_S), =_S) \xrightarrow{[h']_\equiv} (\mathscr{P}(\tau_{S'}), =_{S'})$$

Equilogical spaces and T_0-spaces

The assignment

$$(\mathscr{P}(\tau_S), =_S) \xrightarrow{[h']_\equiv} (\mathscr{P}(\tau_{S'}), =_{S'})$$

extends to a full embedding of categories

$$\begin{array}{ccc}
\mathcal{Top}_0 & \overset{\mathbf{Y}}{\hookrightarrow} & \mathcal{Equ} \\
S & \longmapsto & (\mathscr{P}(\tau_S), =_S) \\
h \downarrow & h \longmapsto [h']_\equiv & \downarrow \\
S' & \longmapsto & (\mathscr{P}(\tau_{S'}), =_{S'})
\end{array}$$

The embedding preserve all limits of $\mathcal{Top}_0$ and every exponential which exists in $\mathcal{Top}_0$.

Equilogical spaces and algebraic lattices

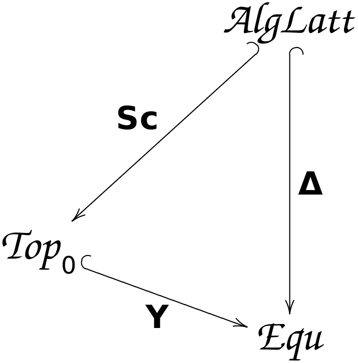
Taking the diagonal relation on an algebraic lattice $\mathscr{A}$ determines an equilogical space. Indeed there is a full embedding

$$\begin{array}{ccc}
\mathcal{AlgLatt} & \overset{\mathbf{\Delta}}{\hookrightarrow} & \mathcal{Equ} \\
\mathscr{A} & \longmapsto & (\mathscr{A}, =_{|\mathscr{A}|}) \\
\downarrow h & h \longmapsto [h]_{\equiv} & \downarrow \\
\mathscr{A}' & \longmapsto & (\mathscr{A}', =_{|\mathscr{A}'|})
\end{array}$$

An algebraic lattice is also a T_0-space when endowed with the Scott topology.

$$\begin{array}{ccc}
\mathcal{AlgLatt} & \overset{\mathbf{Sc}}{\hookrightarrow} & \mathcal{Top}_0 \\
\mathscr{A} & \longmapsto & (|\mathscr{A}|, \sigma) \\
\downarrow h & h \longmapsto h & \downarrow \\
\mathscr{A}' & \longmapsto & (|\mathscr{A}'|, \sigma')
\end{array}$$

Equilogical spaces, T_0-spaces, algebraic lattices



The image of **Y** consists of those equilogical spaces $\mathbf{E} = (\mathscr{A}_{\mathbf{E}}, \smile_{\mathbf{E}})$ where $\smile_{\mathbf{E}} \subseteq |\mathscr{A}_{\mathbf{E}}| \times |\mathscr{A}_{\mathbf{E}}|$ is *subidentical*, *i.e.* $\smile_{\mathbf{E}} \subseteq =_{|\mathscr{A}_{\mathbf{E}}|}$.

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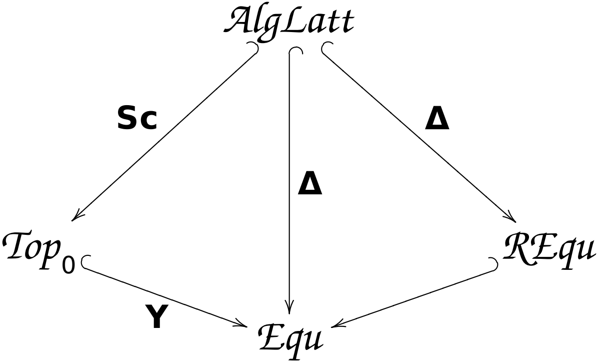
Equilogical spaces, T_0-spaces, algebraic lattices



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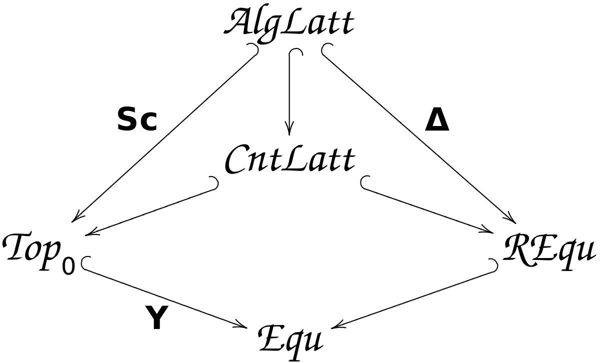
$\mathcal{REqu}$: the full subcategory on the reflexive equilogical spaces.

Equilogical spaces, T_0-spaces, algebraic lattices



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The monad Σ^2

The Sierpinski space $\Sigma =$  is an algebraic lattice.

Consider the self-adjunction

$$\mathcal{E}qu^{\mathrm{op}} \underset{\mathbf{\Delta}(\Sigma)^{(-)}}{\overset{\mathbf{\Delta}(\Sigma)^{(-)}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{E}qu$$

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and the induced (strong) monad Σ^2 on $\mathcal{E}qu$

$$\mathcal{E}qu \xrightarrow{\Sigma^{(\Sigma^{(-)})}} \mathcal{E}qu \qquad \eta_{\mathbf{E}} := \mathbf{E} \xrightarrow{\widehat{\mathrm{ev} \circ \mathrm{swap}}} \Sigma^{(\Sigma^{\mathbf{E}})}$$

$$\mu_{\mathbf{E}} := \Sigma^{(\Sigma^{(\Sigma^{(\Sigma^{\mathbf{E}})})})} \xrightarrow{\Sigma^{\eta_{\Sigma^{\mathbf{E}}}}} \Sigma^{(\Sigma^{\mathbf{E}})}$$

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Thielecke, H.
Continuation Semantics and Self-adjointness. *Electr.Not.Theor.Comp.Sci.* 1997

The monad Σ^2

Σ has a canonical structure of Σ^2-algebra.

Any power $\Sigma^{\mathbf{E}}$ of Σ has the pointwise structure of Σ^2-algebra.

Any algebra for the monad Σ^2 has a canonically induced order from the retraction pair

$$\mathbf{A} \xrightarrow{\eta_{\mathbf{A}}} \Sigma^{(\Sigma^{\mathbf{A}})} \xrightarrow{\alpha} \mathbf{A}, \qquad \alpha \circ \eta_{\mathbf{A}} = \mathrm{id}_{\mathbf{A}}$$

The monad Σ^2 and soberification

For $\mathbf{E} = (\mathscr{A}_{\mathbf{E}}, \smile_{\mathbf{E}})$ the exponential $\Sigma^{\mathbf{E}}$ is $(\Sigma^{\mathscr{A}_{\mathbf{E}}}, \smile_{\Sigma^{\mathbf{E}}})$

where, for $\mathscr{A}_{\mathbf{E}} \underset{g}{\overset{f}{\rightrightarrows}} \Sigma$ Scott-continuous,

$$f \smile_{\Sigma^{\mathbf{E}}} g \text{ if and only if, whenever } a \smile_{\mathbf{E}} b, \text{ it is } f(a) = g(b).$$

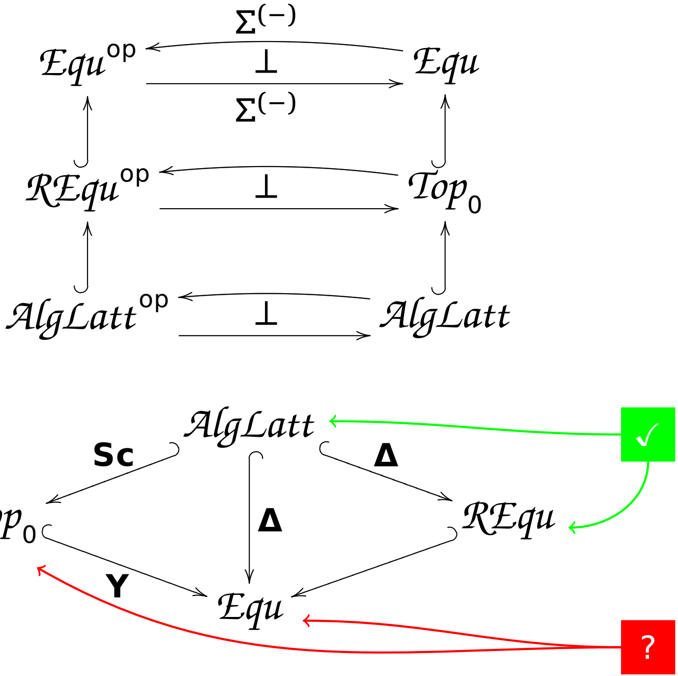
If $\smile_{\mathbf{E}}$ is subreflexive, then $\smile_{\Sigma^{\mathbf{E}}}$ is reflexive.
If $\smile_{\mathbf{E}}$ is reflexive, then $\smile_{\Sigma^{\mathbf{E}}}$ is subreflexive.

For a T_0-space S

- the equilogical space $\Sigma^{(\Sigma^S)}$ is a T_0-space
- the equilogical space $[\Sigma^S, \Sigma]$ of Σ^2-homomorphisms from Σ^S to Σ is the soberification of the space S

Bucalo, A., Rosolini, G.
Sobriety for equilogical spaces. Theoret. Comput. Sci. 546 (2014)

What are the algebras for Σ^2?



The monad Σ^2 and frames

A *frame* is a complete lattice such that finite meets distribute over arbitrary joins:

$$x \wedge \bigvee Y = \bigvee \{x \wedge y \mid y \in Y\}.$$

$\mathcal{Frm}$: the category of frames and frames homomorphisms.

Because the frame structure on Σ is continuous,
the global section functor lifts along the forgetful functors:

$$\begin{array}{ccc}
\mathcal{Equ}^{\Sigma^2} & \xrightarrow{\quad \Gamma \quad} & \mathcal{Frm} \\
{\scriptstyle \mathbf{U}}\downarrow & & \downarrow{\scriptstyle \mathrm{U}} \\
\mathcal{Equ} & \xrightarrow{\quad \Gamma \quad} & \mathcal{Set}
\end{array}$$

The monad Σ^2 and frames

$F = (|F|, \leq)$: a frame

$\mathscr{I}_F$: the algebraic lattice of *ideals of* F, *i.e.*
non-empty, downward-closed, upward-directed subsets of $|F|$.

$\mathscr{I}_F$ is

(♣) the free directed-complete poset on the poset $(|F|, \leq)$

(♠) the free frame on the bounded distributive lattice $(|F|, \leq)$.

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Since $\mathscr{I}_F$ is (♠), consider the frame structure map

$$\bigvee^{\uparrow} : \mathscr{I}_F \to F.$$

Let $\sim_F$ be the equivalence relation on $\mathscr{I}_F$ obtained by taking the kernel pair of the function $\bigvee^{\uparrow}$:

$$I \sim_F J \text{ if and only if } \bigvee^{\uparrow} I = \bigvee^{\uparrow} J.$$

Consider the reflexive equilogical space

$$\mathbf{L}(F) := (\mathscr{I}_F, \sim_F).$$

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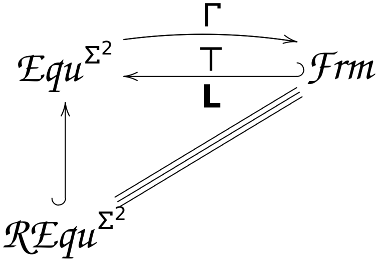

Frames are the Σ^2-algebras on the reflexive equilogical spaces.

Sketch of the proof

Every object $\mathbf{E}$ in $\mathcal{E}qu$ appears in a quotient of the form

$$\mathbf{E} \twoheadleftarrow S \hookrightarrow \mathscr{A}$$

The equilogical space underlying a Σ^2-algebra $\Sigma^{(\Sigma^{\mathbf{A}})} \xrightarrow{\alpha} \mathbf{A}$ appears in a quotient of the form

$$\mathbf{A} \twoheadleftarrow Q \hookrightarrow \mathscr{F}$$

subframe

preserves $\bigwedge$ and $\bigvee$

algebraic frame

$$\Gamma(\mathbf{A}) \twoheadleftarrow \Gamma(Q) \hookrightarrow \Gamma(\mathscr{F})$$

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For any frame homomorphism h

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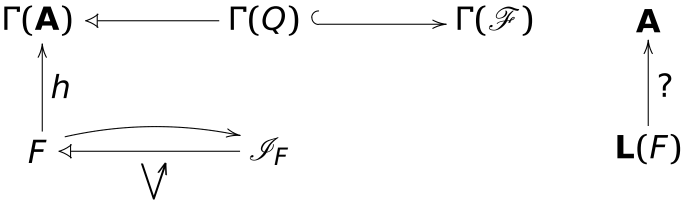

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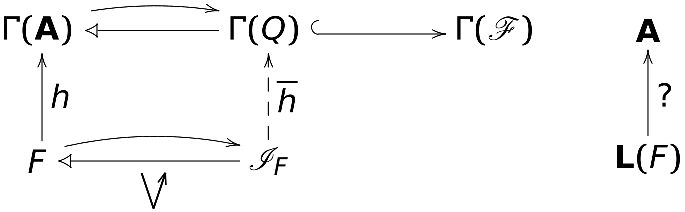

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$$\begin{array}{ccccc} \Gamma(\mathbf{A}) & \underset{\vartriangleleft\!\!\longleftarrow}{\longrightarrow} & \Gamma(Q) & \hookrightarrow & \Gamma(\mathscr{F}) \\ \uparrow h & & \uparrow \overline{h} & & \\ F & \underset{\bigvee}{\overset{\longrightarrow}{\vartriangleleft\!\!\longleftarrow}} & \mathscr{I}_F & & \end{array} \qquad \begin{array}{c} \mathbf{A} \\ \uparrow \left[\overline{h}\right]_{\equiv} \\ \mathbf{L}(F) \end{array}$$

Frames as topological algebras

$$\mathcal{Frm}$$

$$\cong$$

$$\mathcal{REqu}^{\Sigma^2}$$

$$\cong$$

$$\left(\left(\Sigma^{(\Sigma^{\mathcal{REqu}})}\right)_{\text{split}}\right)^{\Sigma^2}$$

$$\cong$$

$$\left(\left(\Sigma^{\mathcal{Top}_0}\right)_{\text{split}}\right)^{\Sigma^2}$$

$$\cong$$

$$\left(\left(\left(\mathcal{Top}_0\right)_{\Sigma^2}\right)^{\text{op}}_{\text{split}}\right)^{\Sigma^2}$$

Power, J., Robinson, E.P.
Premonoidal categories and notions of computation. Math. Struc. Comput. Sci. 7 (1997)