Every Sufficiently Cohesive topos is infinitesimally generated

M. Menni

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and
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History of the PSSL?

"The PSSL grew out of Dana Scott's seminars for his research students at Oxford. Initially it took place at Oxford, Cambridge and Sussex, usually in buildings left locked and vacant over the weekend. It never asked for or received any official recognition or funding. Lectures were informal." [From G. C. Wraith's webpage]

First Meeting: Oxford, 1–2 May 1976

Friday
14.00: Gonzalo Reyes (Montréal) Coherent logic
16.00: Roy Dyckhoff (St Andrews) Sheaves and bundles
Saturday
09.00: Robert Seely (Cambridge) Hyperdoctrines and proof theory
10.00: Wilfrid Hodges (Bedford College) Uniform reduction for local functors
11.00: Harold Simmons (Aberdeen) j-maps on Heyting algebras
11.45: Mike Fourman (Oxford) Sober spaces in topoi
14.15: Martin Hyland (Oxford) Postprandial discursion
15.15: Peter Johnstone (Cambridge) Some short theorems on Kuratowski-finiteness
16.45: Robin Grayson (Oxford) Finiteness in intuitionistic set theory
Sunday
09.30: Gavin Wraith (Sussex) Etale spectrum as a classifying topos
10.30: Julian Cole (Sussex) The bicategory of topoi
11.45: Gonzalo Reyes (Montréal) Negations of coherent formulae

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$$S = 2 = \{\bullet, \bullet\} \longmapsto \begin{tikzcd} \bullet \arrow[r, shift left] & \bullet \arrow[l, shift left] \end{tikzcd}$$

4. (Nullstellensatz) There is an epi natural transformation $\theta : p_* G \to p_! G$. ('Every piece has a point'.)

Leibniz graphs

Definition

An object G in $\widehat{M}$ is called Leibniz if $\theta : p_* G \to p_! G$ is an iso.

Intuition:

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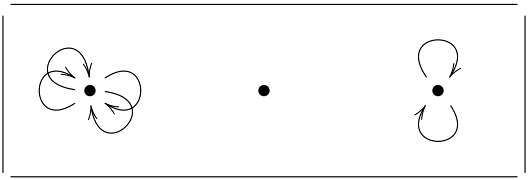

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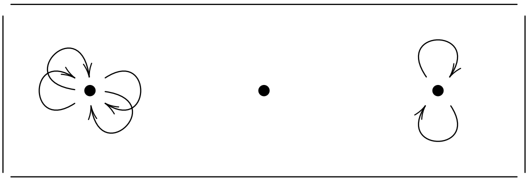

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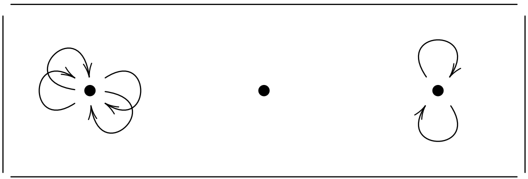

Let $s^* : \mathcal{L} \to \widehat{M}$ be the full subcategory of Leibniz graphs.

Lemma

$\mathcal{L}$ is a topos and $s^* : \mathcal{L} \to \widehat{M}$ is the inverse image of an essential geometric morphism $s : \widehat{M} \to \mathcal{L}$.

$\widehat{M}$ and $\mathcal{L}$

$$\begin{tikzcd}
\widehat{M} \arrow[rr, bend left, "s"] \arrow[d, shift right=6, "p_!"'] \arrow[d, phantom, shift right=4, "\dashv"] \arrow[d, phantom, shift right=2, "\dashv"] \arrow[d, phantom, shift left=2, "\dashv"] \arrow[d, shift left=2, "p_*"] & & \mathcal{L} \arrow[d, shift right=6, "q_!"'] \arrow[d, shift left=2, "q_*"] \\
\mathbf{Set} \arrow[u, shift right=2, "p^*"'] \arrow[u, shift left=6, "p^!"'] & & \mathbf{Set} \arrow[u, shift right=2, "q^*"'] \arrow[u, shift left=6, "q^!"']
\end{tikzcd}$$

$$p_! \dashv p^* \dashv p_* \dashv p^! \qquad q_! \dashv q^* \dashv q_* \dashv q^!$$

but one important difference:

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\mathbf{Set} & & \mathbf{Set}
\end{tikzcd}
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but one important difference:

$$p^! 2 \text{ connected} \qquad \text{vs} \qquad \theta : q_* \to q_! \text{ iso.}$$

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$$\begin{tikzcd}[column sep=large, row sep=large]
\widehat{M} \arrow[rr, bend left, "s"] \arrow[d, shift right=9, "p_!"'] \arrow[d, shift right=3, phantom, "\dashv"] \arrow[d, shift left=3, "p_*"] \arrow[d, shift left=6, phantom, "\dashv"] & & \mathcal{L} \arrow[d, shift right=9, "q_!"'] \arrow[d, shift left=3, "q_*"] \\
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Proposition (Lawvere'07)

The smallest subtopos of $\widehat{M}$ containg $s^* : \mathcal{L} \to \widehat{M}$ is $\widehat{M}$.

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$$\begin{tikzcd}[column sep=huge]
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Proof.

$\mathcal{L} = \widehat{N}$ for monoid N.

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Proof.

$\mathcal{L} = \widehat{N}$ for monoid N. Let $L = s^*(N(\_, *))$. Observe that $M(\_, *) = \begin{tikzcd} \bullet \arrow[r, shift left] \arrow[loop below] & \bullet \arrow[l, shift left] \arrow[loop below] \end{tikzcd}$ is a retract of L^L. $\square$

Axiomatic Cohesion

"An explicit science of cohesion is needed to account for the varied background models for dynamical mathematical theories. Such a science needs to be sufficiently expressive to explain how these backgrounds are so different from other mathematical categories, and also different from one another and yet so united that they can be mutually transformed."

F. W. Lawvere. Axiomatic Cohesion, TAC 2007.

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& \mathcal{E}_{\neg\neg} \arrow[u, shift right=3, "p^*"'] \arrow[u, shift left=9, "p^!"'] &
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$\mathcal{E}$ will be called sufficiently cohesive if $p^! 2$ is connected.

Examples

Proposition (Johnstone'11 + Lawvere · Menni'15)

If $\mathcal{C}$ is a small category with terminal and every object has a point then $\mathcal{E} = \widehat{\mathcal{C}}$ is a topos of spaces and $\mathcal{E}_{\neg\neg} = \mathbf{Set}$.

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In this case, $\mathcal{E}$ is sufficiently cohesive iff some object of $\mathcal{C}$ has at least two points.

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For example: simplicial sets [EZ], cubical sets [Kan], the classifier of non-trivial boolean algebras [Lawvere], Ball complexes [RoyThesis], many Gaeta toposes (e.g. for the theory of distributive lattices or for that of $\mathbb{C}$-algebras), ...

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Other examples over **Set**: Zariski topos (over $\mathbb{C}$), sheaves for sites of monoids of 'tame' endos of $[0, 1]$. In this cases, $\mathcal{E}_{\neg\neg} = \mathbf{Set}$.

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Other examples: Variants of Zariski (over other fields, say $\mathbb{R}$) where $\mathcal{E}_{\neg\neg} \neq \mathbf{Set}$.

Leibniz objects

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Since Ω is injective, it is a retract of J^J. $\square$

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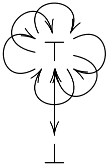

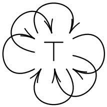

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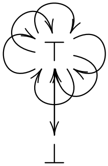

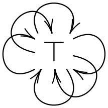

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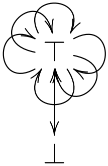

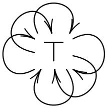

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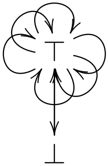

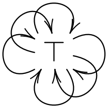

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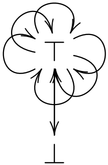

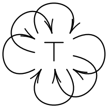

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Substantiality implies that the transposition $\Omega \to J^J$ is mono. $\square$

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Exercise

Why do the above equivalent conditions hold if $\mathcal{C}$ has terminal object, every object has a point and some object has at least two distinct points?

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Sufficient cohesion implies that $\top : 1 \to J$ is substantial. $\square$

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