Cotorsion theories in locally presentable abelian categories

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A full subcategory $\mathcal{A}$ of a category $\mathcal{K}$ is called *weakly coreflective* if each object K in $\mathcal{K}$ has a weak coreflection, i.e., a morphism $c_K : K^* \to K$ where K^* is in $\mathcal{A}$ such that every morphism $f : A \to K$ with A in $\mathcal{A}$ factorizes (not necessarily uniquely) through c_K.

$\mathcal{A}$ is *stably weakly coreflective* if, in addition, any morphism $f : K^* \to K^*$ with $c_K \cdot f = c_K$ is an isomorphism.

A stable weak coreflection is unique up to an isomorphism.

Every weakly coreflective subcategory closed under retracts is closed under coproducts.

Projective R-modules form a weakly coreflective subcategory of R-modules which is stably weakly coreflective iff projective R-modules are closed under directed colimits, i.e., iff they coincide with flat R-modules. It contrasts to the fact that injective R-modules are always stably weakly reflective in R-modules.

Enochs (1981) conjectured that flat modules are always stably weakly coreflective in R-modules.

Enochs' conjecture was proved by Bican, El Bashir and Enochs in 2001. Later, El Bashir generalized the proof to any Grothendieck category (2006).

Theorem 1. (El Bashir) Let $\mathcal{A}$ be an accessible full subcategory of a Grothendieck category closed under coproducts and directed colimits. Then $\mathcal{A}$ is stably weakly coreflective.

Flat modules satisfy these assumptions in R-modules.
Grothendieck categories are precisely locally presentable abelian categories where filtered colimits are exact. Every locally finitely presentable abelian category is Grothendieck. Our main result is that exactness of filtered colimits can be avoided.

Theorem 2. Let $\mathcal{A}$ be an accessible full subcategory of a locally presentable abelian category closed under coproducts and directed colimits. Then $\mathcal{A}$ is stably weakly coreflective.

Under Vopěnka's principle, any full subcategory of a locally presentable category closed under directed colimits is accessible.

The following result was proved by Enochs (1981) for Grothendieck categories and I realized in 2004 that it is true in any locally finitely presentable category.

Proposition 1. Let $\mathcal{K}$ be a locally presentable category and $\mathcal{A}$ its weakly coreflective full subcategory closed under directed colimits. Then $\mathcal{A}$ is stably weakly coreflective.

A category $\mathcal{K}$ *has enough λ-pure quotients* if for each object K there is, up to isomorphism, only a set of morphisms $f : L \to K$ such that $f = h \cdot g$, g λ-pure epimorphism implies that g is an isomorphism.

Any dual of a locally λ-presentable category has enough λ-pure quotients.

The key part of the proof of the flat cover conjecture is that any abelian locally presentable category has enough λ-pure quotients for some regular cardinal λ.

Proposition 2. Let $\mathcal{K}$ be a locally presentable category having enough λ-pure quotients for some regular cardinal λ. Let $\mathcal{A}$ be an accessible full subcategory of $\mathcal{K}$ which is closed under coproducts and directed colimits. Then $\mathcal{A}$ is weakly coreflective in $\mathcal{K}$.

Locally presentable abelian categories appear as categories of algebras over *additive* accessible monads $T : \mathbf{Set} \to \mathbf{Set}$.

T is additive if it contains abelian group operations $+, -, 0$ and accessible if it preserves κ-filtered colimits for some κ.

As examples, we get contramodules over topological rings (Positselski 2010). If R is a complete, separated topological ring where open right ideals form a base of neighbourhoods of 0 then the monad for contramodules sends a set S to the set $R[[S]]$ of formal linear combinations of elements of S with the coeficients converging to 0.

Contramodules provide a large class of locally presentable abelian categories with enough projective objects which are not necessarily Grothendieck. This happens, for the topological ring $\mathbb{Z}_p$ of p-adic integers or for the topological ring $k[[z]]$ of formal power series over a field k.

Contramodules were introduced originally by Eilenberg and Moore (1965) in the case of coalgebras over commutative rings as module-like objects dual-analogous to, but different from, the more familiar comodules. They were also studied by Barr (1970).

Flat R-modules are defined using the tensor product and they are characterized as directed colimits of projective modules. Positselski (2010) introduced flat R-contramodules over a topological ring R using a contratensor product with discrete R-modules. Any projective contramodule is flat.

Proposition 3. The category of flat R-contramodules is accessible and closed under directed colimits and coproducts in R-contramodules.

Corollary 1. Flat R-contramodules are stably weakly coreflective in R-contramodules.

There is not known whether any flat R-contramodule is a directed colimit of projective R-contramodules.

Let $\mathcal{K}$ be an abelian category. A pair $(\mathcal{A}, \mathcal{B})$ of classes of objects of $\mathcal{K}$ is called a *cotorsion theory* if

$$\mathcal{B} = \{B \mid \mathrm{Ext}(A, B) = 0 \quad \text{for all} \quad A \in \mathcal{A}\} \quad \text{and}$$
$$\mathcal{A} = \{A \mid \mathrm{Ext}(A, B) = 0 \quad \text{for all} \quad B \in \mathcal{B}\}\,.$$

An $\mathcal{A}$-monomorphism is defined as a monomorphism whose cokernel belongs to $\mathcal{A}$. Dually, a $\mathcal{B}$-epimorphism is an epimorphism whose kernel is in $\mathcal{B}$. The class of all $\mathcal{A}$-monomorphisms is denoted by $\mathcal{A}$-Mono. Dually, $\mathcal{B}$-Epi.

$\mathcal{B}$ coincides with objects injective to $\mathcal{A}$-monomorphisms and $\mathcal{A}$ with objects projective to $\mathcal{B}$-epimorphisms. Thus $\mathcal{A}$ contains all projective objects and $\mathcal{B}$ contains all injective objects.

Lemma 1. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion theory in an abelian category $\mathcal{K}$ such that any object of $\mathcal{K}$ is a quotient of an object from $\mathcal{A}$ and a subobject of an object from $\mathcal{B}$. Then $\mathcal{A}\text{-Mono}^{\square} = \mathcal{B}\text{-Epi}$ and ${}^{\square}\mathcal{B}\text{-Epi} = \mathcal{A}\text{-Mono}$.

A cotorsion theory $(\mathcal{A}, \mathcal{B})$ is called *complete* if any object K has a weak coreflection $c_K : K^* \to K$ to $\mathcal{A}$ which is a $\mathcal{B}$-epimorphism and a weak reflection $r_K : K \to K_*$ to $\mathcal{B}$ which is an $\mathcal{A}$-monomorphism.

A cotorsion theory $(\mathcal{A}, \mathcal{B})$ is *generated* by a set $\mathcal{S}$ if

$$\mathcal{B} = \{B \mid \mathrm{Ext}(S, B) = 0 \quad \text{for all} \quad S \in \mathcal{S}\}$$

Proposition 4. Let $\mathcal{K}$ be a locally presentable abelian category and $(\mathcal{A}, \mathcal{B})$ a cotorsion theory generated by a set. Suppose that any object of $\mathcal{K}$ is a quotient of an object from $\mathcal{A}$ and a subobject of an object from $\mathcal{B}$. Then $(\mathcal{A}\text{-Mono}, \mathcal{B}\text{-Epi})$ is a weak factorization system.

Corollary 2. Let $\mathcal{K}$ be a locally presentable abelian category and $(\mathcal{A}, \mathcal{B})$ a cotorsion theory generated by a set. Suppose that any object of $\mathcal{K}$ is a quotient of an object from $\mathcal{A}$ and a subobject of an object from $\mathcal{B}$. Then $(\mathcal{A}, \mathcal{B})$ is complete.

It suffices the take weak factorizations $0 \to K^* \to K$ and $K \to K_* \to 0$.

Corollary 3. Let $\mathcal{K}$ be a locally presentable abelian category and $(\mathcal{A}, \mathcal{B})$ a complete cotorsion theory in $\mathcal{K}$ such that $\mathcal{A}$ is closed under directed colimits. Then $\mathcal{A}$ is stably weakly coreflective.

We say that $\mathcal{A}$ is *strongly closed under directed colimits* in $\mathcal{K}$ if $\mathcal{A}$-monomorphisms are closed under directed colimits in any comma-category $K \downarrow \mathcal{K}$. It implies that $\mathcal{A}$ is closed under directed colimits. In a Grothendieck category, the two concepts are equivalent.

Corollary 4. Let $\mathcal{K}$ be a locally presentable abelian category and $(\mathcal{A}, \mathcal{B})$ a complete cotorsion theory in $\mathcal{K}$ such that $\mathcal{A}$ is strongly closed under directed colimits. Then $\mathcal{B}$ is stably weakly reflective in $\mathcal{K}$.

The assumptions of Proposition 4 are satisfied in any Grothendieck category having enough projectives (because it has enough injectives). In particular, in R-modules.

R-contramodules have enough projectives but not enough injectives. Thus the assumption about $\mathcal{B}$ is not automatic.

Like in modules, an R-contramodule is called *cotorsion* if it is Ext-orthogonal to flat R-contramodules.

Theorem 3. Flat and cotorsion R-contramodules form a complete cotorsion theory in R-contramodules.