The Gabriel dimension of a module M has nothing to do with the category $\mathcal{M}\mathrm{od}$-R in which it lives.

It is determined by the lattice $\mathcal{S}\mathrm{ub}(M)$ of its submodules.

The construction of this dimension is a particular example of a more general lattice theoretic technique.

There are other application of this method, such as the not so well known Boyle dimension of a module.

There are also non-module applications, such as the Cantor-Bendixson rank of a topological space, and more generally of a frame.

The two module examples

We work with the family $\mathbb{B}(R)$ of 'basic' classes of modules.

We use certain quotients, localizations, of $\mathcal{M}$od-R

$$\mathcal{M}\text{od-}R/\boldsymbol{\mathcal{T}}$$

each of which is determined by a

$$\text{Division class} = \boldsymbol{\mathcal{T}} = \text{Hereditary torsion class}$$

$\mathbb{B}(R)$ carries a divisional closure. $\mathbb{DVS} : \mathbb{B}(R) \longrightarrow \mathbb{B}(R)$

To measure relative to $\boldsymbol{\mathcal{T}}$ we select a larger class of modules

(Gabriel) $\boldsymbol{\mathcal{T}}$-simple $\boldsymbol{\mathcal{S}}$ $\boldsymbol{\mathcal{T}}$-complemented (Boyle)

and extend to a division class to produce a closure operation.

$$\mathbb{GAB} = \mathbb{DVS} \circ \mathbb{SMP} \qquad \mathbb{DVS} \circ \mathbb{CMP} = \mathbb{BOY}$$

We iterate to produce the measuring filtration relative to $\boldsymbol{\mathcal{T}}$.

$$\boldsymbol{\mathcal{T}}_0 = \boldsymbol{\mathcal{T}} \subseteq \boldsymbol{\mathcal{T}}_1 \subseteq \boldsymbol{\mathcal{T}}_2 \subseteq \cdots \subseteq \boldsymbol{\mathcal{T}}_\alpha \subseteq \cdots$$

Global inflators and closure operations

A G-inflator $\boldsymbol{inf}_\bullet$ attaches to each module M an inflator $\boldsymbol{inf}_M$ on $\mathcal{S}\mathrm{ub}(M)$. For example, the socle operator done properly.

Such inflators must interact with morphism $N \longrightarrow M$ sensibly.

There are correspondences with more standard gadgets.

G-inflator $\longleftrightarrow$ Pre-hereditary torsion class = Supine class

G-closure $\longleftrightarrow$ Hereditary torsion class = Division class

These are easier to deal with than the standard gadgets.

G-inflator = global pre-nucleus G-closure = global nucleus

This idea was produced by K.L. Chew in 1965.

Idioms

An idiom is a complete lattice Λ such that

$$a \wedge \bigvee X = \bigvee \{a \wedge x \mid x \in X\}$$

for all *directed* subsets X.

Egs. For a module M the lattice $\mathcal{S}\mathrm{ub}(M)$ is a modular idiom.
A frame is a distributive idiom.
Each complete boolean algebra is a complemented frame.

Each appropriate quotient of Λ is given by a nucleus, a closure operation j such that

$$j(a \wedge b) = j(a) \wedge j(b)$$

By iterating a pre-nucleus (a non-idempotent nucleus) we generate a nucleus.

The length of the iteration is some kind of measure.

Idiom gadgetry

We work with the family $\mathbb{B}(\Lambda)$ of 'basic' sets of intervals.
To produce a quotient of Λ we collapse a basic set $\mathcal{B}$, and more.
A division set $\mathcal{D}$ corresponds to a quotient.

$$\begin{array}{cccccc} j & \text{Nucleus} & \longleftrightarrow & \text{Division set} & \mathcal{D} \\ \boldsymbol{inf} & \text{Pre-nucleus} & \longleftrightarrow & \text{Supine set} & \mathcal{S} \end{array}$$

Each basic set $\mathcal{B}$ has a divisional closure $\mathbb{D}\mathrm{vs}(\mathcal{B})$

Each division set $\mathcal{D}$ has larger supine set $\mathbb{S}\mathrm{mp}(\mathcal{D})$ $\mathbb{C}\mathrm{mp}(\mathcal{D})$
These give $\mathbb{G}\mathrm{ab} = \mathbb{D}\mathrm{vs} \circ \mathbb{S}\mathrm{mp}$ $\mathbb{B}\mathrm{oy} = \mathbb{D}\mathrm{vs} \circ \mathbb{C}\mathrm{mp}$
which we can iterate to produce measures of Λ.

We can also work with pre-nuclei. When j corresponds to $\mathcal{D}$

$$\begin{array}{ccccccc} \mathbb{S}\mathrm{mp}(\mathcal{D}) & \longleftrightarrow & \boldsymbol{soc}_j & \longmapsto & \boldsymbol{soc}_j^{\infty} = \boldsymbol{Gab}(j) & \longleftrightarrow & \mathbb{G}\mathrm{ab}(\mathcal{D}) \\ \mathbb{C}\mathrm{mp}(\mathcal{D}) & \longleftrightarrow & \boldsymbol{cbd}_j & \longmapsto & \boldsymbol{cbd}_j^{\infty} = \boldsymbol{Boy}(j) & \longleftrightarrow & \mathbb{B}\mathrm{oy}(\mathcal{D}) \end{array}$$

The slicing trick

We slice a class $\boldsymbol{\mathcal{B}} \in \mathbb{B}(R)$ of modules by a module M to produce a set $\mathbb{B}(\Lambda)$ of intervals of $\Lambda = \mathcal{S}\mathrm{ub}(M)$.

$$B/A \in \boldsymbol{\mathcal{B}} \quad \Longleftrightarrow \quad [A, B] \in \big\langle M \big\rangle(\boldsymbol{\mathcal{B}})$$

$$\begin{array}{ccc}
\mathbb{B}(R) & \xrightarrow{\quad \langle M \rangle(\cdot) \quad} & \mathbb{B}(\Lambda) \\
\mathbb{OPR} \big\downarrow & \langle M \rangle \circ \mathbb{OPR} = \mathbb{O}\mathrm{pr} \circ \langle M \rangle & \big\downarrow \mathbb{O}\mathrm{pr} \\
\mathbb{B}(R) & \xrightarrow{\quad \langle M \rangle(\cdot) \quad} & \mathbb{B}(\Lambda)
\end{array}$$

$$\mathbb{DVS}, \mathbb{D}\mathrm{vs}, \quad \mathbb{SMP}, \mathbb{S}\mathrm{mp}, \quad \mathbb{CMP}, \quad \mathbb{C}\mathrm{mp}, \quad \mathbb{GAB}, \mathbb{G}\mathrm{ab}, \quad \mathbb{BOY}, \mathbb{B}\mathrm{oy}$$

For $\boldsymbol{\mathcal{T}}, M$ with $\mathcal{D} = \big\langle M \big\rangle(\boldsymbol{\mathcal{T}})$ $\quad \boldsymbol{soc}_{\boldsymbol{\mathcal{T}},M} = \boldsymbol{soc}_{\mathcal{D}}$ $\ \boldsymbol{cbd}_{\boldsymbol{\mathcal{T}},M} = \boldsymbol{cbd}_{\mathcal{D}}$

Various measuring gadgets can be produced in a lattice theoretic context, and then the measure of a module is a particular case of this general version.