# The Gabriel dimension of a module M has nothing to do with the category $\mathcal{M}od$ -R in which it lives.

It is determined by the lattice Sub(M) of its submodules.

The construction of this dimension is a particular example of a more general lattice theoretic technique.

There are other application of this method, such as the not so well known Boyle dimension of a module.

There are also non-module applications, such as the Cantor-Bendixson rank of a topological space, and more generally of a frame.

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#### The two module examples

We work with the family  $\mathbb{B}(R)$  of 'basic' classes of modules. We use certain quotients, localizations, of  $\mathcal{M}$ od-R

#### $\mathcal{M}$ od- $R/\mathcal{T}$

each of which is determined by a

Division class  $= \mathcal{T} =$  Hereditary torsion class  $\mathbb{B}(R)$  carries a divisional closure.  $\mathbb{DVS} : \mathbb{B}(R) \longrightarrow \mathbb{B}(R)$ To measure relative to  $\mathcal{T}$  we select a larger class of modules (Gabriel)  $\mathcal{T}$ -simple  $\mathcal{S}$   $\mathcal{T}$ -complemented (Boyle) and extend to a division class to produce a closure operation.  $\mathbb{GAB} = \mathbb{DVS} \circ \mathbb{SMP}$   $\mathbb{DVS} \circ \mathbb{CMP} = \mathbb{BOY}$ We iterate to produce the measuring filtration relative to  $\mathcal{T}$ .

$${\mathcal{T}}_0 = {\mathcal{T}} \subseteq {\mathcal{T}}_1 \subseteq {\mathcal{T}}_2 \subseteq \cdots \subseteq {\mathcal{T}}_\alpha \subseteq \cdots$$

#### Global inflators and closure operations

A G-inflator  $inf_{\bullet}$  attaches to each module M an inflator  $inf_M$ on Sub(M). For example, the socle operator done properly. Such inflators must interact with morphism  $N \longrightarrow M$  sensibly. There are correspondences with more standard gadgets.

These are easier to deal with than the standard gadgets.

 $G-inflator = global pre-nucleus \qquad G-closure = global nucleus$ 

This idea was produced by K.L. Chew in 1965.

## Idioms

An idiom is a complete lattice  $\Lambda$  such that

$$a \land \bigvee X = \bigvee \{a \land x \mid x \in X\}$$

for all *directed* subsets X.

Egs. For a module M the lattice Sub(M) is a modular idiom. A frame is a distributive idiom.

Each complete boolean algebra is a complemented frame.

Each appropriate quotient of  $\Lambda$  is given by a nucleus, a closure operation j such that

$$j(a \wedge b) = j(a) \wedge j(b)$$

By iterating a pre-nucleus (a non-idempotent nucleus) we generate a nucleus.

The length of the iteration is some kind of measure.

## Idiom gadgetry

We work with the family  $\mathbb{B}(\Lambda)$  of 'basic' sets of intervals. To produce a quotient of  $\Lambda$  we collapse a basic set  $\mathcal{B}$ , and more. A division set  $\mathcal{D}$  corresponds to a quotient.

j	Nucleus	$\longleftrightarrow$	Division set	$\mathcal{D}$
inf	Pre-nucleus	$\longleftrightarrow$	Supine set	${\mathcal S}$

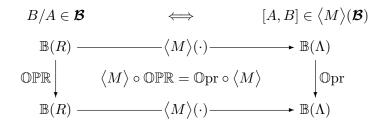
Each basic set  $\mathcal{B}$  has a divisional closure  $\mathbb{D}vs(\mathcal{B})$ 

Each division set  $\mathcal{D}$  has larger supine set  $\operatorname{Smp}(\mathcal{D})$   $\operatorname{Cmp}(\mathcal{D})$ These give  $\operatorname{Gab} = \mathbb{D}vs \circ \operatorname{Smp}$   $\mathbb{B}oy = \mathbb{D}vs \circ \mathbb{C}mp$ which we can iterate to produce measures of  $\Lambda$ .

We can also work with pre-nuclei. When j corresponds to  $\mathcal{D}$ 

### The slicing trick

We slice a class  $\mathcal{B} \in \mathbb{B}(R)$  of modules by a module M to produce a set  $\mathbb{B}(\Lambda)$  of intervals of  $\Lambda = Sub(M)$ .



 $\mathbb{DVS}$ ,  $\mathbb{Dvs}$ ,  $\mathbb{SMP}$ ,  $\mathbb{Smp}$ ,  $\mathbb{CMP}$ ,  $\mathbb{Cmp}$ ,  $\mathbb{GAB}$ ,  $\mathbb{Gab}$ ,  $\mathbb{BOY}$ ,  $\mathbb{Boy}$ For  $\mathcal{T}$ , M with  $\mathcal{D} = \langle M \rangle(\mathcal{T})$   $soc_{\mathcal{T},M} = soc_{\mathcal{D}} \ cbd_{\mathcal{T},M} = cbd_{\mathcal{D}}$ 

Various measuring gadgets can be produced in a lattice theoretic context, and then the measure of a module is a particular case of this general version.