Pullbacks of Essential Inclusions of Grothendieck Toposes

Guilherme Frederico Lima

University of Cambridge

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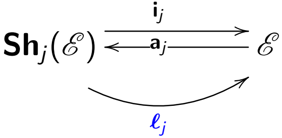
Essential Inclusions

An inclusion of toposes is *essential* if the inverse image functor has a left adjoint:



Theorem: (Kelly & Lawvere - 1989):

An inclusion of Grothendieck toposes $\mathbf{Sh}_j(\mathscr{E}) \hookrightarrow \mathscr{E}$ is essential iff each element of the generating set $\mathcal{G}$ of $\mathscr{E}$ has a smallest dense subobject.

$$\sigma_X \rightarrowtail X \in \mathcal{G}$$

$$\sigma : \mathcal{G} \to \mathscr{E}$$

$$\sigma : \mathcal{G}^{op} \times \mathcal{G} \to \mathbf{Set}$$

Corollary: An inclusion of Grothendieck toposes $\mathbf{Sh}_j(\mathscr{E}) \hookrightarrow \mathscr{E}$ is essential iff the closure operation

$$\mathbf{cl}_j : \mathbf{Sub} \to \mathbf{Sub}$$

has a left adjoint

$$\mathbf{int}_j : \mathbf{Sub} \to \mathbf{Sub}.$$

$$\mathbf{int}_j(X) = \mathbf{im}(\sigma \otimes X \to X)$$

Theorem: (Kelly & Lawvere - 1989)

Let $\mathbb{C}$ be a small category. There is an order preserving bijection between essential inclusions into $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ and idempotent ideals on $\mathbb{C}$.

$\mathcal{I} \subseteq \mathbf{Mor}(\mathbb{C})$ is an *ideal* if:

$$f \in \mathcal{I} \Rightarrow fg \in \mathcal{I} \text{ and } f \in \mathcal{I} \Rightarrow hf \in \mathcal{I}.$$

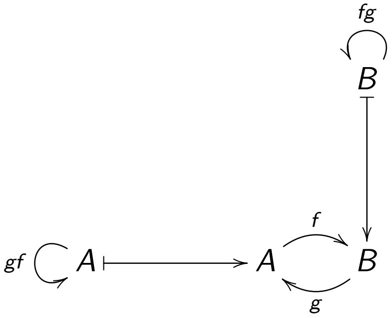
$$\mathcal{I} : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Set}$$

$\mathcal{I} \subseteq \mathbf{Mor}(\mathbb{C})$ is *idempotent* if:

$$f \in \mathcal{I} \Rightarrow f = gh \;\text{ where } g, h \in \mathcal{I}.$$

Are essential inclusions always stable under pullback? **NO!**

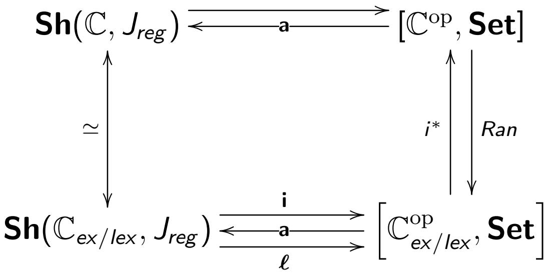
First example (Kelly & Lawvere): Let $\mathbb{C}$ be the free category generated by the graph with two objects A, B, and two morphisms $f : A \to B$ and $g : B \to A$.



Now take presheaves on these categories...

Second example (Karazeris): Let $\mathbb{C}$ be a regular category and $\mathbb{C}_{ex/lex}$ be its ex/lex completion. Let J_{reg} the singleton-epi topology.

The Comparison Lemma gives us $\mathbf{Sh}(\mathbb{C}, J_{reg}) \simeq \mathbf{Sh}(\mathbb{C}_{ex/lex}, J_{reg})$. Thus we have



When is the pullback of an essential inclusion also an essential inclusion?

Motivating example

$$T : \mathbb{C} \to \mathbb{D}$$

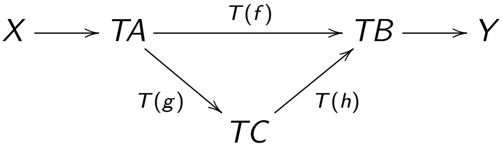
$$\begin{array}{ccc} & & [\mathbb{D}^{\mathrm{op}}, \mathbf{Set}] \\ & & Lan \uparrow \;\; \downarrow T^{*} \\ \mathbf{Sh}(\mathbb{C}, J) & \underset{\boldsymbol{\ell}}{\overset{\mathbf{i}}{\rightarrow}} \; \overset{\mathbf{a}}{\leftarrow} & [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}] \end{array}$$

In this situation we can apply T to $\sigma : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Set}$.

We have $\langle T(\sigma) \rangle$:

$$X \longrightarrow TA \xrightarrow{T(f)} TB \longrightarrow Y$$

where $f \in \sigma$.



$g, h, \in \sigma$.

Thus we have

$$
\begin{array}{ccc}
\mathbf{Sh}(\mathbb{D}, T(J)) & \begin{array}{c} \xrightarrow{\ \mathbf{i}'\ } \\ \xleftarrow{\ \mathbf{a}'\ } \\ \xrightarrow[\ \boldsymbol{\ell}'\ ]{} \end{array} & [\mathbb{D}^{\mathrm{op}}, \mathbf{Set}] \\
 & & Lan \uparrow \quad \downarrow T^{*} \\
\mathbf{Sh}(\mathbb{C}, J) & \begin{array}{c} \xrightarrow{\ \mathbf{i}\ } \\ \xleftarrow{\ \mathbf{a}\ } \\ \xrightarrow[\ \boldsymbol{\ell}\ ]{} \end{array} & [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]
\end{array}
$$

How do we generalise this approach?

Theorem: Let $\mathscr{E}$ be a Grothendieck topos with generating set $\mathcal{G}$. There is an order preserving bijection between essential inclusions into $\mathscr{E}$ and subfunctors of the Yoneda embedding $\sigma \rightarrowtail y : \mathcal{G} \to \mathscr{E}$ such that

$$\sigma \otimes \sigma \to \sigma$$

is an epi.

$$\sigma \otimes \sigma(A, B) \twoheadrightarrow \sigma \cdot \sigma(A, B) \rightarrowtail \sigma(A, B) \rightarrowtail y(A, B)$$

$$[f, g]/\sim \qquad [f, g]/\sim' \qquad gf \qquad gf$$

Theorem: Let $\mathbb{L}$ be a locale. There is an order preserving bijection between *cartesian* essential inclusions into $\mathbf{Sh}(\mathbb{L})$ and cartesian subfunctors of the Yoneda embedding $\sigma \rightarrowtail y : \mathbb{L} \to \mathbf{Sh}(\mathbb{L})$ such that

$$\sigma \otimes \sigma \cong \sigma.$$

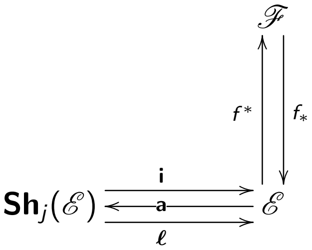
Theorem: Let $\mathscr{E}$ be a Grothendieck topos. There is an order preserving bijection between essential inclusions into $\mathscr{E}$ and endofunctors $\textbf{int} : \mathscr{E} \to \mathscr{E}$ such that

$$\textbf{int} \rightarrowtail \mathrm{id},$$

$$\textbf{int} \circ \textbf{int} \cong \textbf{int},$$

and **int** preserves epis and small coproducts.

The general case



$$X$$

$$f_*(X) \qquad f^*\mathbf{int}(f_*(X)) \rightarrowtail$$

$$f^*f_*(X) \qquad f^*\mathbf{int}(f_*(X)) \rightarrowtail f^*f_*(X) \to$$

$$X \qquad \mathbf{im}(f^*\mathbf{int}(f_*(X)) \rightarrowtail f^*f_*(X) \to$$

$$X)\mathbf{int}_f(X) = \mathbf{im}(f^*\mathbf{int}(f_*(X)) \rightarrowtail f^*f_*(X) \to X)$$

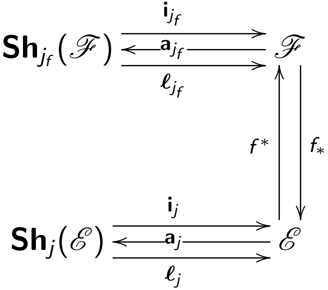
Theorem: Let $f : \mathscr{F} \to \mathscr{E}$ be a geometric morphism between Grothendieck toposes. If f_* preserves epis and small coproducts, then we have a functor between the partial orders of essential inclusions into $\mathscr{E}$ and essential inclusions into $\mathscr{F}$:

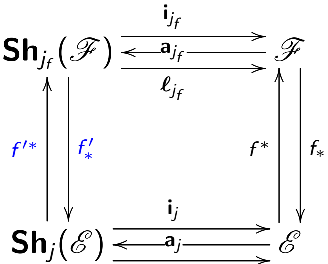
$$\mathbf{EssInc}(\mathscr{E}) \to \mathbf{EssInc}(\mathscr{F}),$$

which sends $\mathbf{int} : \mathscr{E} \to \mathscr{E}$ to $\mathbf{int}_f : \mathscr{F} \to \mathscr{F}$

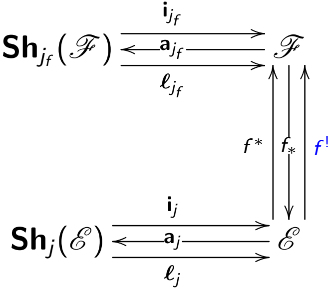
If f_* preserves epis and small coproducts

If $f^*\mathbf{int}(\eta) : f^*\mathbf{int} \to f^*\mathbf{int} f_* f^*$ is an epi



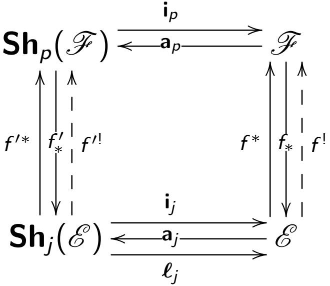


$f_* $ preserves epis and small coproducts iff $f_* \dashv f^!$.



$\mathbf{i}_{j_f}$

Looking at the pullback



$\mathbf{Sh}_p(\mathscr{F}) \simeq \mathbf{Sh}_{j_f}(\mathscr{F})$ iff f_* preserves discrete objects and f'_* is faithful.

Thank you!

References

1. G.M. Kelly, F.W. Lawvere - On The Complete Lattice Of Essential Localizations. *Bull. Soc. Math. Belg. Ser. A, XLI(2): 289-319, 1989*

2. G.F. Lima - Cartesian and Finite-Product-Preserving Essential Inclusions of Grothendieck Toposes. *Ph.D. Thesis, 2016*