

Pullbacks of Essential Inclusions of Grothendieck Toposes

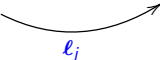
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Essential Inclusions

An inclusion of toposes is *essential* if the inverse image functor has a left adjoint:

$$\mathbf{Sh}_j(\mathcal{E}) \begin{array}{c} \xrightarrow{\mathbf{i}_j} \\ \xleftarrow{\mathbf{a}_j} \end{array} \mathcal{E}$$


The diagram shows the relationship between the topos of sheaves $\mathbf{Sh}_j(\mathcal{E})$ and the topos \mathcal{E} . The forward arrow is labeled \mathbf{i}_j and the backward arrow is labeled \mathbf{a}_j . A curved arrow labeled ℓ_j indicates a left adjoint from $\mathbf{Sh}_j(\mathcal{E})$ to \mathcal{E} .

Theorem: (Kelly & Lawvere - 1989):

An inclusion of Grothendieck toposes $\mathbf{Sh}_j(\mathcal{E}^o) \hookrightarrow \mathcal{E}^o$ is essential iff each element of the generating set \mathcal{G} of \mathcal{E}^o has a smallest dense subobject.

$$\sigma_X \mapsto X \in \mathcal{G}$$

$$\sigma : \mathcal{G} \rightarrow \mathcal{E}^o$$

$$\sigma : \mathcal{G}^{op} \times \mathcal{G} \rightarrow \mathbf{Set}$$

Corollary: An inclusion of Grothendieck toposes $\mathbf{Sh}_j(\mathcal{E}) \hookrightarrow \mathcal{E}$ is essential iff the closure operation

$$\mathbf{cl}_j : \mathbf{Sub} \rightarrow \mathbf{Sub}$$

has a left adjoint

$$\mathbf{int}_j : \mathbf{Sub} \rightarrow \mathbf{Sub}.$$

$$\mathbf{int}_j(X) = \mathbf{im}(\sigma \otimes X \rightarrow X)$$

Theorem: (Kelly & Lawvere - 1989)

Let \mathbb{C} be a small category. There is an order preserving bijection between essential inclusions into $[\mathbb{C}^{\text{op}}, \mathbf{Set}]$ and idempotent ideals on \mathbb{C} .

$\mathcal{I} \subseteq \mathbf{Mor}(\mathbb{C})$ is an *ideal* if:

$$f \in \mathcal{I} \Rightarrow fg \in \mathcal{I} \text{ and } f \in \mathcal{I} \Rightarrow hf \in \mathcal{I}.$$

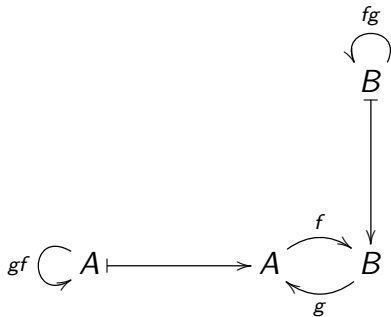
$$\mathcal{I} : \mathbb{C}^{\text{op}} \times \mathbb{C} \rightarrow \mathbf{Set}$$

$\mathcal{I} \subseteq \mathbf{Mor}(\mathbb{C})$ is *idempotent* if:

$$f \in \mathcal{I} \Rightarrow f = gh \text{ where } g, h \in \mathcal{I}.$$

Are essential inclusions always stable under pullback? **NO!**

First example (Kelly & Lawvere): Let \mathbb{C} be the free category generated by the graph with two objects A, B , and two morphisms $f : A \rightarrow B$ and $g : B \rightarrow A$.



Now take presheaves on these categories...

Second example (Karazeris): Let \mathbb{C} be a regular category and $\mathbb{C}_{ex/lex}$ be its *ex/lex* completion. Let J_{reg} the singleton-epi topology.

The Comparison Lemma gives us $\mathbf{Sh}(\mathbb{C}, J_{reg}) \simeq \mathbf{Sh}(\mathbb{C}_{ex/lex}, J_{reg})$. Thus we have

$$\begin{array}{ccc}
 \mathbf{Sh}(\mathbb{C}, J_{reg}) & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\mathbf{a}} \\ \xrightarrow{\quad} \end{array} & [\mathbb{C}^{\text{op}}, \mathbf{Set}] \\
 \updownarrow \simeq & & \updownarrow \begin{array}{c} i^* \\ \text{Ran} \end{array} \\
 \mathbf{Sh}(\mathbb{C}_{ex/lex}, J_{reg}) & \begin{array}{c} \xrightarrow{\mathbf{i}} \\ \xleftarrow{\mathbf{a}} \\ \xrightarrow{\ell} \end{array} & [\mathbb{C}_{ex/lex}^{\text{op}}, \mathbf{Set}]
 \end{array}$$

When is the pullback of an essential inclusion also an essential inclusion?

Motivating example

$$T : \mathbb{C} \rightarrow \mathbb{D}$$

$$\begin{array}{ccc} & & [\mathbb{D}^{\text{op}}, \mathbf{Set}] \\ & & \uparrow \text{Lan} \\ & & \downarrow T^* \\ \mathbf{Sh}(\mathbb{C}, J) & \begin{array}{c} \xrightarrow{\mathbf{i}} \\ \xleftarrow{\mathbf{a}} \\ \xrightarrow{\ell} \end{array} & [\mathbb{C}^{\text{op}}, \mathbf{Set}] \end{array}$$

In this situation we can apply T to $\sigma : \mathbb{C}^{\text{op}} \times \mathbb{C} \rightarrow \mathbf{Set}$.

We have $\langle T(\sigma) \rangle$:

$$X \longrightarrow TA \xrightarrow{T(f)} TB \longrightarrow Y$$

where $f \in \sigma$.

$$\begin{array}{ccccc} X & \longrightarrow & TA & \xrightarrow{T(f)} & TB & \longrightarrow & Y \\ & & \searrow^{T(g)} & & \nearrow^{T(h)} & & \\ & & & TC & & & \end{array}$$

$g, h, \in \sigma$.

Thus we have

$$\begin{array}{ccc} \mathbf{Sh}(\mathbb{D}, T(J)) & \begin{array}{c} \xrightarrow{i'} \\ \xleftarrow{a'} \\ \xrightarrow{\ell'} \end{array} & [\mathbb{D}^{\text{op}}, \mathbf{Set}] \\ & & \begin{array}{c} \uparrow \text{Lan} \\ \downarrow T^* \end{array} \\ \mathbf{Sh}(\mathbb{C}, J) & \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{a} \\ \xrightarrow{\ell} \end{array} & [\mathbb{C}^{\text{op}}, \mathbf{Set}] \end{array}$$

How do we generalise this approach?

Theorem: Let \mathcal{E} be a Grothendieck topos with generating set \mathcal{G} . There is an order preserving bijection between essential inclusions into \mathcal{E} and subfunctors of the Yoneda embedding

$$\sigma \mapsto y : \mathcal{G} \rightarrow \mathcal{E} \text{ such that}$$

$$\sigma \otimes \sigma \rightarrow \sigma$$

is an epi.

$$\sigma \otimes \sigma(A, B) \twoheadrightarrow \sigma \cdot \sigma(A, B) \twoheadrightarrow \sigma(A, B) \twoheadrightarrow y(A, B)$$

$$[f, g] / \sim$$

$$[f, g] / \sim'$$

$$gf$$

$$gf$$

Theorem: Let \mathbb{L} be a locale. There is an order preserving bijection between *cartesian* essential inclusions into $\mathbf{Sh}(\mathbb{L})$ and cartesian subfunctors of the Yoneda embedding

$\sigma \mapsto y : \mathbb{L} \rightarrow \mathbf{Sh}(\mathbb{L})$ such that

$$\sigma \otimes \sigma \cong \sigma.$$

Theorem: Let \mathcal{E} be a Grothendieck topos. There is an order preserving bijection between essential inclusions into \mathcal{E} and endofunctors $\mathbf{int} : \mathcal{E} \rightarrow \mathcal{E}$ such that

$$\mathbf{int} \succrightarrow \text{id},$$

$$\mathbf{int} \circ \mathbf{int} \cong \mathbf{int},$$

and \mathbf{int} preserves epis and small coproducts.

The general case

$$\begin{array}{ccc}
 & & \mathcal{F} \\
 & & \uparrow f^* \\
 & & \downarrow f_* \\
 \text{Sh}_j(\mathcal{E}) & \begin{array}{c} \xrightarrow{\mathbf{i}} \\ \xleftarrow{\mathbf{a}} \\ \xrightarrow{\ell} \end{array} & \mathcal{E}
 \end{array}$$

$$\begin{array}{l}
 f_*(X) \\
 f^*f_*(X) \\
 X \\
 X) \mathbf{int}_f(X) = \mathbf{im}(f^*\mathbf{int}(f_*(X)) \rightrightarrows f^*f_*(X) \rightarrow X)
 \end{array}
 \qquad
 \begin{array}{l}
 X \\
 f^*\mathbf{int}(f_*(X)) \rightrightarrows \\
 f^*\mathbf{int}(f_*(X)) \rightrightarrows f^*f_*(X) \rightarrow \\
 \mathbf{im}(f^*\mathbf{int}(f_*(X)) \rightrightarrows f^*f_*(X) \rightarrow \\
 \mathbf{im}(f^*\mathbf{int}(f_*(X)) \rightrightarrows f^*f_*(X) \rightarrow X)
 \end{array}$$

Theorem: Let $f : \mathcal{F} \rightarrow \mathcal{E}$ be a geometric morphism between Grothendieck toposes. If f_* preserves epis and small coproducts, then we have a functor between the partial orders of essential inclusions into \mathcal{E} and essential inclusions into \mathcal{F} :

$$\mathbf{EssInc}(\mathcal{E}) \rightarrow \mathbf{EssInc}(\mathcal{F}),$$

which sends $\mathbf{int} : \mathcal{E} \rightarrow \mathcal{E}$ to $\mathbf{int}_f : \mathcal{F} \rightarrow \mathcal{F}$

If f_* preserves epis and small coproducts

If $f^* \mathbf{int}(\eta) : f^* \mathbf{int} \rightarrow f^* \mathbf{int} f_* f^*$ is an epi

$$\begin{array}{ccc}
 \mathbf{Sh}_{j_f}(\mathcal{F}) & \begin{array}{c} \xrightarrow{i_{j_f}} \\ \xleftarrow{a_{j_f}} \\ \xrightarrow{l_{j_f}} \end{array} & \mathcal{F} \\
 & & \uparrow f^* \downarrow f_* \\
 \mathbf{Sh}_j(\mathcal{E}) & \begin{array}{c} \xrightarrow{i_j} \\ \xleftarrow{a_j} \\ \xrightarrow{l_j} \end{array} & \mathcal{E}
 \end{array}$$

$$\begin{array}{ccc}
 \mathbf{Sh}_{j_f}(\mathcal{F}) & \begin{array}{c} \xrightarrow{i_{j_f}} \\ \xleftarrow{a_{j_f}} \\ \xrightarrow{l_{j_f}} \end{array} & \mathcal{F} \\
 \uparrow f'^* \downarrow f'_* & & \uparrow f^* \downarrow f_* \\
 \mathbf{Sh}_j(\mathcal{E}) & \begin{array}{c} \xrightarrow{i_j} \\ \xleftarrow{a_j} \\ \xrightarrow{l_j} \end{array} & \mathcal{E}
 \end{array}$$

f_* preserves epis and small coproducts iff $f_* \dashv f^!$.

$$\begin{array}{ccc}
 \mathbf{Sh}_{j_f}(\mathcal{F}) & \begin{array}{c} \xrightarrow{i_{j_f}} \\ \xleftarrow{a_{j_f}} \\ \xrightarrow{\ell_{j_f}} \end{array} & \mathcal{F} \\
 & & \begin{array}{c} \uparrow f^* \\ \downarrow f_* \\ \uparrow f^! \end{array} \\
 \mathbf{Sh}_j(\mathcal{C}) & \begin{array}{c} \xrightarrow{i_j} \\ \xleftarrow{a_j} \\ \xrightarrow{\ell_j} \end{array} & \mathcal{C} \\
 & & \downarrow i_{j_f}
 \end{array}$$

Looking at the pullback

$$\begin{array}{ccc}
 \mathbf{Sh}_p(\mathcal{F}) & \begin{array}{c} \xrightarrow{i_p} \\ \xleftarrow{a_p} \end{array} & \mathcal{F} \\
 \begin{array}{c} \uparrow f'^* \\ \downarrow f'_* \\ \uparrow f'^! \\ \downarrow f'^! \end{array} & & \begin{array}{c} \uparrow f^* \\ \downarrow f_* \\ \uparrow f^! \\ \downarrow f^! \end{array} \\
 \mathbf{Sh}_j(\mathcal{E}) & \begin{array}{c} \xrightarrow{i_j} \\ \xleftarrow{a_j} \\ \xrightarrow{\ell_j} \end{array} & \mathcal{E}
 \end{array}$$

$\mathbf{Sh}_p(\mathcal{F}) \simeq \mathbf{Sh}_j(\mathcal{F})$ iff f_* preserves discrete objects and f'_* is faithful.

Thank you!

References

1. G.M. Kelly, F.W. Lawvere - On The Complete Lattice Of Essential Localizations. *Bull. Soc. Math. Belg. Ser. A*, *XLI(2): 289-319, 1989*
2. G.F. Lima - Cartesian and Finite-Product-Preserving Essential Inclusions of Grothendieck Toposes. *Ph.D. Thesis, 2016*