Infinitesimal Models of Algebraic Theories

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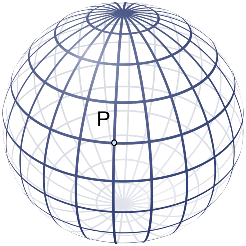
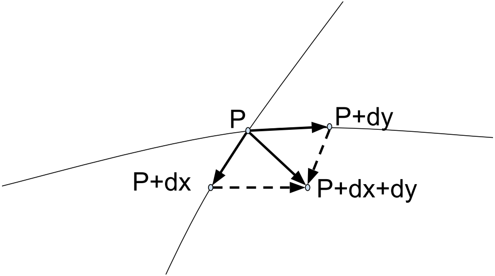
Intuition: Manifolds are infinitesimally affine





- Manifolds become linear at each point *P* when considering an *infinitely small neighbourhood of P*
- *'become linear'* means that the linear operations hold within some piece of the manifold itself, not elsewhere
- Abandoning the base point dependency: A manifold is infinitesimally affine

Some facts in Synthetic Differential Geometry (SDG)

Let R be an *an* $\mathbb{R}$*-algebra of line type* (in a topos $\mathbb{S}$), i.e. R satisfies the *Kock-Lawvere Axiom*. Recall that

- A Weil-algebra (over $\mathbb{R}$) is a local, finite dimensional $\mathbb{R}$-algebra. The maximal ideal of W is nilpotent.
- Infinitesimal neighbourhoods of 0 are (by definition) the R-spectra D_W of Weil-algebras W
- The Kock-Lawvere axiom (K-L) states that any map $f : D_W \to R$ is a polynomial map, and the polynomial is determined uniquely by f. The degree and form of the polynomial is determined by W.
- $D(n) := \{(d_1, \ldots, d_n) \in R^n \mid d_i d_j = 0,\ 1 \leq i, j \leq n\}$, then $D(n) \cong D_{W(n,2)}$ for the Weil-algebra $W(n, 2) = \mathbb{R}[X_1, \ldots, X_n]/(\{X_i X_j \mid 1 \leq i, j \leq n\})$
- $D = D(1) = \{d \in R \mid d^2 = 0\}$ is D_W for $W = \mathbb{R}[X]/(X^2)$ the ring of dual numbers.

How does SDG reflect the intuition about manifolds?

- For $(P, Q) \in R^n$ we say that P and Q are (infinitesimal) (1-)neighbours, '$P \sim Q$', when $(P - Q) \in D(n)$
- The reflexive, symmetric relation $R^n_{\langle 1 \rangle} = \{(P, Q) \in R^n \times R^n \mid P \sim Q\}$ is called the first neighbourhood of the diagonal in R^n
- We define $R^n_{\langle k \rangle} = \{(P_0, \ldots, P_k) \in R^{k+1} \mid P_i \sim P_j \; 1 \leq i, j \leq k\}$

R^n is infinitesimally affine

For any $(\lambda_0, \ldots, \lambda_k) \in R^{k+1}$, $\sum_{i=0}^{k} \lambda_i = 1$ and $\langle P_0, \ldots, P_k \rangle \in R^n_{\langle k \rangle}$ the affine combination $\sum_{i=0}^{k} \lambda_i P_i$ is a neighbour of every P_j

- Due to K-L every map $f : R^n \to R^m$ preserves the neighbourhood relation and such affine combinations

Formally open subobjects of R^n are infinitesimally affine

- A subobject $U \to M$ is said to be *formally open*, if the subsequent square is a pullback for any Weil-algebra W

$$\begin{array}{ccc} U^{D_W} & \xrightarrow{\mathrm{ev}_0} & U \\ \downarrow & & \downarrow \\ M^{D_W} & \xrightarrow{\mathrm{ev}_0} & M \end{array}$$

- A formally open subobject $U \to R^n$ is closed under the neighbour relation, i.e., $P \in U$ and $P \sim Q$ for a $Q \in R^n$ implies $Q \in U$
- U is infinitesimally affine in the same sense as R^n
- Let $U \to R^n$, $V \to R^m$ be formally open subobjects. Every map $f : U \to V$ preserves the neighbour relation and affine combinations of mutual neighbours.

Formal manifolds are infinitesimally affine

- A space M is said to be a *(formal) n-manifold* if there is a jointly epimorphic family of formally open subobjects $U_i \to M$, $i \in I$, such that each U_i is (isomorphic to) a formally open subobject $U_i \to R^n$
- We can define a neighbourhood relation $\sim$ on M and $M_{\langle k \rangle} = \{(P_0, \dots, P_k) \in M^{k+1} \mid P_i \sim P_j \ 1 \leq i, j \leq k\}$

Theorem (A. Kock)

Let M be a formal manifold.

(1) *There is a (in a certain sense, unique) neighbourhood relation $M_{\langle 1 \rangle}$ on M, such that for any $(\lambda_0, \dots, \lambda_k) \in R^k$, $\sum_{i=0}^{k} \lambda_i = 1$ and $\langle P_0, \dots, P_k \rangle \in M_{\langle k \rangle}$ the affine combination $\sum_{i=0}^{k} \lambda_i P_i$ exists in M and is a neighbour of every P_j.*

(2) *Every map $f : M \to N$ between formal manifolds preserves the neighbourhood relation and affine combinations of mutual neighbours.*

Questions

- Kock's result shows that any formal manifold is infinitesimally affine, and hence infinitesimally linear at each point
- However, there are some questions left

Questions

In what sense is a manifold a model of the theory of affine spaces over R? What is the structure that would define an infinitesimally affine space?

More questions

Can we make precise what it means for a smooth space to be infinitesimally a model of a certain geometry?

Clones

Let C be a category with finite limits. We shall represent algebraic theories by *clones*.

Definition

Clone A clone O over C has the following data:

(1) for every $n \in \mathbb{N}$ an object $O(n)$ (of *n-ary operations*),

(2) for every $n, k \in \mathbb{N}$ a map $*_{nk} : O(n) \times O(k)^n \to O(k)$ (*substitution*),

(3) for every $n \geq 1$ and $1 \leq j \leq n$ global elements $\pi^n_j : 1 \to O(n)$ (*projections*),

and axioms:

(1) (associativity) for every $f \in O(n)$, $g_1, \ldots, g_n \in O(m)$, $h_1, \ldots h_m \in O(k)$

$$\begin{aligned} & f *_{nk} (g_1 *_{mk} (h_1, \ldots, h_m)), \ldots, g_n *_{mk} (h_1, \ldots, h_m)) \\ & = (f *_{nm} (g_1, \ldots, g_n)) *_{mk} (h_1, \ldots, h_m) \end{aligned}$$

(2) (projection) for every $n \geq 1$, $1 \leq j \leq n$, $f_1, \ldots, f_n \in O(n)$

$$\pi^n_i *_{nm} (f_1, \ldots, f_n) = f_i$$

(3) (unit) for each $f \in O(n)$, $f *_{nn} (\pi^n_1, \ldots, \pi^n_n) = f$

Examples of clones

(1) The clone $\mathcal{A}_R$ of affine combinations over R in $\mathbb{S}$.

- $\mathcal{A}_R(n) = \{(\lambda_1, \ldots, \lambda_n) \in R^n \mid \sum_{i=1}^{k} \lambda_i = 1\}$, $\mathcal{A}_R(0) = \emptyset$
- $(\lambda_1, \ldots, \lambda_n) *_{nk} (\overrightarrow{\mu}_1, \ldots, \overrightarrow{\mu}_n) = \sum_{i=1}^{n} \lambda_i \overrightarrow{\mu}_i$

(2) The clone $\mathcal{L}_R$ of linear combinations over R in $\mathbb{S}$.

- $\mathcal{L}_R(n) = R^n$
- $(\lambda_1, \ldots, \lambda_n) *_{nk} (\overrightarrow{\mu}_1, \ldots, \overrightarrow{\mu}_n) = \sum_{i=1}^{n} \lambda_i \overrightarrow{\mu}_i$

(3) The clone $\mathcal{G}$ for groups in Set.

- $\mathcal{G}(n) = F_G[X_1, \ldots, X_n]$ (free group of n-generators)
- $*_{nk}$ is substitution of terms

Infinitesimal algebras of a clone

Let O be a clone in a category C with finite limits.

- Idea from SDG: the clone O can only operate on the subobjects of points that are mutual neighbours!

Definition

An *infinitesimal algebra* (IA) M of a clone O has the following data

(1) for each $n \in \mathbb{N}$ a subobject $A_{\langle n \rangle} \hookrightarrow A^n$ (infinitesimal structure)

(2) for each $n \in \mathbb{N}$ a map $\alpha_n : O(n) \times A_{\langle n \rangle} \to A_{\langle 1 \rangle}$ (action)

and satisfies the axioms

(1) $A_{\langle 1 \rangle} = A$, ~~$A_{\langle 0 \rangle} \cong 1$~~,

(2) (projection) for $n \geq 1$, $1 \leq j \leq n$,

$$\alpha_n(\pi^n_j, \langle P_1, \ldots, P_n \rangle) = P_j$$

(3) (neighbourhood) let $f^1, \ldots, f^k \in O(n)$, $\langle P_1, \ldots, P_n \rangle \in A_{\langle n \rangle}$, then $\langle \alpha_n(f^1, \langle P_1, \ldots, P_n \rangle), \ldots, \alpha_n(f^n, \langle P_1, \ldots, P_n \rangle) \rangle \in A_{\langle k \rangle}$

(4) (associativity) for every $f \in O(n)$, $g_1, \ldots, g_n \in O(m)$, $\langle P_1, \ldots P_m \rangle \in A_{\langle m \rangle}$

$$\begin{aligned} &\alpha_n(f, \langle \alpha_m(g_1, \langle P_1, \ldots, P_m \rangle), \ldots, \alpha_m(g_n, \langle P_1, \ldots, P_m \rangle) \rangle) \\ &= \alpha_m((f *_{nm} (g_1, \ldots, g_n)), \langle P_1, \ldots, P_m \rangle) \end{aligned}$$

Some basic observations and examples

Let A be a O-IA.

- $A_{\langle 2 \rangle}$ is a reflexive, symmetric relation '$\sim$'.
- Each $A_{\langle n \rangle}$, $n \geq 2$ is symmetric and the diagonal $\Delta_n : A \to A^n$ factors through A
- $O(0)^n \to A^n$ factors through $A_{\langle n \rangle}$
- For $\langle P_1, \ldots, P_n \rangle \in A_{\langle n \rangle}$ we have $P_i \sim P_j$. But for $(Q_i, \ldots, Q_n) \in A^n$, $Q_i \sim Q_j$ doesn't necessarily imply $\langle Q_1, \ldots, Q_n \rangle \in A_{\langle n \rangle}$. If this holds then the infinitesimal structure on A is generated by '$\sim$'.

- Every (total) O-algebra A is an IA with $A_{\langle n \rangle} = A^n$
- $O(n)$ is a (total) O-algebra

More examples

- R^n with the nil-square infinitesimal structure induced by $\sim$ is an IA for $\mathcal{A}_R$
  - Essentially, we only need to show the neighbourhood axiom. This follows from the following fact: if $P_i \sim P_j$ for $1 \leq j \leq n$ and $Q \sim P_i$, $1 \leq i \leq n$, then $Q \sim \sum_{i=1}^{n} \lambda_i P_i$ for $(\lambda_1, \ldots, \lambda_n) \in \mathcal{A}_R(n)$
- Each formally open subobject $U \rightarrow R^n$ is an IA for $\mathcal{A}_R$ with the induced infinitesimal structure from R^n
- Each formal manifold is an IA for $\mathcal{A}_R$
- The infinitesimal space D with infinitesimal structure generated by $D_{\langle 2 \rangle} = D(2)$ is an IA for $\mathcal{L}_R$, and hence for $\mathcal{A}_R$. D is not a formal manifold, since D is not formally open

Properties of the forgetful functor

Let C be a finite limit category and O a clone in C.

- There is an obvious notion of structure preserving morphism of O-IA's
- We can form the category O-IAlg(C) and have a forgetful functor $U : \text{O-IAlg}(C) \to C$

Theorem

- *U is faithful and amnestic, but doesn't reflect identities, in general*
- *U lifts limits (uniquely)*
- *The O-algebra $O(n)$ is initial, and 1 is terminal in O-IAlg(C)*

If C is a Grothendieck topos

- *U lifts (wide) pushouts of total O-algebras (!)*
- *U lifts quotients of admissible congruences*
- *U preserves regular epis*

More properties

Theorem (continued)

- *U lifts filtered colimits*
- *For each $X \in C$ the endofunctor $(-)^X$ lifts via U*
- *If $O(0) = \emptyset$ then U lifts (small) coproducts*
- O-IAlg(C) *is a complete and cocomplete regular category, but is not exact, in general*

- For a clone O in Set one can show easily that the theory of O-IA's is sketchable by a finite limit sketch. This allows us to talk about O-IA's in any category C with finite limits.
- For a clone O in a Grothendieck topos $\mathbb{S}$ the theory of O-IA's is $\mathbb{S}$-sketchable by an $\mathbb{S}$-enriched limit sketch (!)

Theorem?

For a Grothendieck topos $\mathbb{S}$ the category O-IAlg($\mathbb{S}$) is locally presentable.

Infinitesimally Affine Spaces

Let C be a category with finite products and a strict initial object. Let R be a (commutative) ring in C. We can define the clones $\mathcal{A}_R$ and $\mathcal{L}_R$ in C.

Definition (Infinitesimally affine space)

(1) We call an infinitesimal $\mathcal{A}_R$-algebras M in C an *infinitesimally affine* (IA) space, and denote the category of $\mathcal{A}_R$ IA's by $\mathrm{IAff}(C)$

(2) We call an infinitesimal $\mathcal{L}_R$-algebra M an *infinitesimally linear* space

- Note that any infinitesimally linear space M is pointed

The formal duals of C^∞-Rings

- Recall that a C^∞-ring A is a model of the Lawvere theory of smooth functions on $\mathbb{R}^n$, $n \in \mathbb{N}$.
  - It is, in particular, an $\mathbb{R}$-algebra that admits an interpretation $\hat{f} : A^n \to A$ for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ preserving composition and projections.
- Define $\mathrm{Sp} = C^\infty\text{-}\mathrm{Rng}^{op}$ and denote by sA the formal dual of the C^∞-ring A
- Sp is complete and cocomplete, since C^∞-Rng is.
- $R = s\,C^\infty(\mathbb{R})$ is an $\mathbb{R}$-algebra object in Sp

Every space in Sp is infinitesimally affine

Theorem

(1) *Let* $R = s\,\mathrm{C}^\infty(\mathbb{R})$ *have the nil-square infinitesimal structure generated by* $P \sim Q :\Leftrightarrow (P-Q)^2 = 0$, *then* R *is an IA space.*

(2) *Let* $sA \in \mathrm{Sp}$ *have the infinitesimal structure that is initial/weak w.r.t the maps* $sA \to R$, *then*

   (i) sA *is an IA space*

   (ii) *every map in* Sp *is a morphism of IA spaces*

(3) *For any global(!) point* $P : 1 \to sA$ *the infinitesimal neighbourhood of* P, $D(P) = \{Q \in sA \mid P \sim Q\} \hookrightarrow sA$, *is infinitesimally linear.*

Every group object in Sp has an infinitesimal subgroup

Theorem

Let G be a group object in Sp *with neutral element $e : 1 \to G$, then $D(e) \hookrightarrow G$ is an infinitesimal subgroup. In particular, every Lie group $s\,\mathrm{C}^\infty(G)$ has an infinitesimal subgroup.*

Infinitesimally affine spaces in Well-adapted models

- Let *Mf* denote the category of smooth, paracompact manifolds and smooth maps. A well-adapted model $(\mathbb{S}, \iota)$ is a Grothendieck topos $\mathbb{S}$ with full embedding $\iota : Mf \to \mathbb{S}$ such that
  (1) ι preserves 1, transversal pullbacks and maps open covers to jointly epimorphic families
  (2) $\iota(\mathbb{R})$ is of line type
- By a result of Bunge& Dubuc ι extends to a finite limit preserving functor Sp $\to \mathbb{S}$
- Hence we recover in particular that the full subcategory of (classical) manifolds in $\mathbb{S}$ is isomorphic to a full subcategory of IAff($\mathbb{S}$) and that every (classical) Lie group has an infinitesimal subgroup.
- More IA spaces can be constructed from these using the lifting properties of the forgetful functor U. (But some of these constructions don't necessarily give us the results we would have liked!)

Thank you for your attention!