Completeness of infinitary intuitionistic logics

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1. $$\bigwedge_{\alpha<\gamma} \phi_\alpha, \qquad \bigvee_{\alpha<\gamma} \phi_\alpha$$
2. $$\forall \mathbf{x}_\gamma \phi, \qquad \exists \mathbf{x}_\gamma \phi$$

   (where $\mathbf{x}_\gamma = \{x_\alpha : \alpha < \gamma\}$)

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  2. Classical dependent choice up to γ (DC_γ):
  $$\bigwedge_{\alpha<\gamma} \forall_{\beta<\alpha} \mathbf{x}_\beta \exists \mathbf{x}_\alpha \psi_\alpha \to \exists_{\alpha<\gamma} \mathbf{x}_\alpha \bigwedge_{\alpha<\gamma} \psi_\alpha$$
  (for disjoint $\mathbf{x}_\alpha$ and such that no variable in $\mathbf{x}_\alpha$ is free in ψ_β for $\beta < \alpha$).

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  (for disjoint $\mathbf{x}_\alpha$ and such that no variable in $\mathbf{x}_\alpha$ is free in ψ_β for $\beta < \alpha$).
- Completeness theorem proved using Boolean algebraic methods and thus relies heavily in the use of the excluded middle axiom.

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Definition (Makkai)

A κ-regular category is a regular category that has κ-limits (i.e., limits of κ-small diagrams) and satisfies further an exactness property of $\mathcal{S}et$ corresponding to the axioms DC_γ of dependent choice up to γ for each $\gamma < \kappa$.

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Consider a κ-chain in a category $\mathcal{C}$ with κ-limits, i.e., a diagram $\Gamma : \gamma^{op} \to \mathcal{C}$ specified by morphisms $(h_{\beta,\alpha} : C_\beta \to C_\alpha)_{\alpha \leq \beta < \gamma}$ with the following condition:

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We say that the morphisms $h_{\beta,\alpha}$ compose transfinitely, and take the limit projection $f_{\beta,0}$ to be the transfinite composite of $h_{\alpha+1,\alpha}$ for $\alpha < \beta$. Then the exactness condition reads that if all maps $h_{\beta,\alpha}$ are epimorphisms, so is $f_{\beta,0}$. Loosely speaking we say that the transfinite composition of epimorphisms is itself an epimorphism.

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- $P_a \vdash P_b$ for each pair a, b such that a is a successor of b

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We can show, on the other hand, that the hypothesis of weak compactness is enough to derive a completeness theorem for κ-coherent theories of cardinality at most κ with respect to $\mathcal{S}et$-valued models κ-coherent logic then extends geometric logic, for which a completeness theorem in terms of $\mathcal{S}et$-valued models is not possible.

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Definition

A κ-coherent category is a κ-complete coherent category with κ-complete subobject lattices where unions of cardinality less than κ are stable under pullback, and where the transfinite composites of jointly covering κ-families of morphisms form a jointly covering family.

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- The corresponding axiom schema in the κ-coherent logic is the following "transfinite transitivity" rule:

$$\frac{\phi_i \vdash_{\mathbf{y}_i} \bigvee_{j\in\gamma^{\beta+1}, j|_\beta = i} \exists \mathbf{x}_j \phi_j \qquad \beta < \gamma, i \in \gamma^\beta \qquad\qquad \phi_i \dashv\vdash_{\mathbf{y}_i} \bigwedge_{\alpha<\beta} \phi_{i|_\alpha} \qquad \beta < \gamma, \text{ limit } \beta, i \in \gamma^\beta}{\phi_\emptyset \vdash_{\mathbf{y}_\emptyset} \bigvee_{f\in\gamma^\gamma} \exists_{\beta<\gamma} \mathbf{x}_{i|_{\beta+1}} \bigwedge_{\beta<\gamma} \phi_{i|_\beta}}$$

for each cardinal $\gamma < \kappa$, where $\mathbf{y}_i$ is the canonical context of ϕ_i, provided that, for every $i \in \gamma^{\beta+1}$, $FV(\phi_i) = FV(\phi_{i|_\beta}) \cup \mathbf{x}_i$ and $\mathbf{x}_{i|_{\beta+1}} \cap FV(\phi_{i|_\beta}) = \emptyset$ for any $\beta < \gamma$, as well as $FV(\phi_i) = \bigcup_{\alpha<\beta} FV(\phi_{i|_\alpha})$ for limit β. Note that we assume that there is a fixed well-ordering of γ^γ for each $\gamma < \kappa$.

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Moreover, we have, as expected:

Proposition

Let κ be an inaccessible cardinal. If $\mathcal{C}$ is κ-coherent, then $\mathcal{S}h(\mathcal{C}, \tau)$ is κ-coherent.

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So:

- Weak compactness is needed for a completeness theorem with respect to κ-Kripke semantics
- Using a generalization of a theorem of Joyal, we can prove that weak compactness is also a sufficient condition.

Completeness theorems for κ-first-order logic

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Moreover, we have:

Theorem (E., 2016)

Let κ be a weakly compact cardinal. If $\mathcal{C}$ is a κ-coherent, Heyting category of cardinality at most κ, then $ev : \mathcal{C} \to Set^{Mod(\mathcal{C})}$ is a conservative, (κ-coherent) and Heyting functor.

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2. the conservativity of $ev : \mathcal{C} \to Set^{Mod(\mathcal{C})}$ is a $\mathcal{S}et$-valued completeness for κ-coherent theories of cardinality at most κ (this uses weak compactness).
3. if $\mathcal{C}$ is in addition a Boolean category, this is Karp's completeness theorem for κ-first-order classical theories of cardinality at most κ.

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- Use strongly compact cardinals to remove the restriction on the cardinality of the theories
- Study the case of finite-quantifier theories over $\mathcal{L}_{\kappa,\omega}$
- Call κ a *Heyting cardinal* if κ-first-order theories of cardinality strictly less than κ are complete for κ-Kripke semantics. Determine its strength within the large cardinal hierarchy.

Thanks for your attention