Exact completion of path categories

Benno van den Berg
ILLC, University of Amsterdam

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Preliminary remarks

- Contents based a joint paper with Ieke Moerdijk:

  Exact completion of path categories and algebraic set theory. arXiv:1603.02456.

- Work described here was motivated by (homotopy) type theory, but I will not say much about the connection.
- Similar work has been done by Frey, Gran, Vitale, Rosolini and possibly others!

Traditional picture (Joyal, Carboni, ...)

For any cartesian category $\mathcal{C}$ there is an exact category $\mathcal{C}_{ex/lex}$ together with an embedding $Y : \mathcal{C} \to \mathcal{C}_{ex/lex}$ such that composing with Y induces an equivalence

$$\mathrm{EX}(\mathcal{C}_{ex/lex}, \mathcal{D}) \to \mathrm{CART}(\mathcal{C}, \mathcal{D})$$

for any exact category $\mathcal{D}$.

The category $\mathcal{C}_{ex/lex}$ can be constructed very concretely: its objects are pseudo-equivalence relations.

Definition

A *pseudo-equivalence relation* is an arrow $\rho = (\rho_1, \rho_2) : R \to X \times X$ for which there are

1. a map $k : X \to R$ such that $\rho k = \Delta_X$,
2. a map $l : R \to R$ such that $\rho_1 l = \rho_2$ and $\rho_2 l = \rho_1$, and
3. a map $m : R \times_X R \to R$ such that $\rho_1 m = \rho_1$ and $\rho_2 m = \rho_2$.

Traditional picture, continued

A morphism $(X, \rho : R \to X \times X)$ to $(Y, \sigma : S \to Y \times Y)$ in $\mathcal{C}_{ex/lex}$ is an equivalence class of arrows $f : X \to Y$ for which there are arrows $\varphi : R \to S$ such that

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \sigma} \\ X \times X & \xrightarrow[f \times f]{} & Y \times Y \end{array}$$

commutes. Two such maps $f, g : X \to Y$ are equivalent if there is an arrow $H : X \to S$ making

$$\begin{array}{ccc} & & S \\ & \overset{H}{\nearrow} & \downarrow{\scriptstyle \sigma} \\ X & \xrightarrow[(f,g)]{} & Y \times Y \end{array}$$

commute. The embedding $Y : \mathcal{C} \to \mathcal{C}_{ex/lex}$ is obtained by sending X to $(X, \Delta_X : X \to X \times X)$.

A theme in the theory of exact completions

Theorem

$\mathcal{C}_{ex/lex}$ is exact and together with $Y : \mathcal{C} \to \mathcal{C}_{ex/lex}$ it is the exact completion of $\mathcal{C}$.

An important theme in the theory of exact completions is that it improves the properties of the original category. For example, we have:

Theorem (Carboni, Rosolini)

$\mathcal{C}_{ex/lex}$ is locally cartesian closed if and only if $\mathcal{C}$ is weakly locally cartesian closed.

This is reminiscent of *setoids* in type theory. To bring this closer to (homotopy) type theory we introduce *path categories* (short for: *categories with path objects*).

Setting

We work in a category $\mathcal{C}$ with two classes of maps:

- *fibrations*
- *weak equivalences*

Terminology:

- A map which is both a fibration and a weak equivalence will be called an *acyclic fibration*.
- If we can factor the diagonal $B \to B \times B$ as a weak equivalence $r : B \to PB$ followed by a fibration $(s, t) : PB \to B \times B$, then PB is a *path object* for B.

Category with path objects, or path category

Axioms

1. $\mathcal{C}$ has a terminal object 1 and $X \to 1$ is always a fibration.
2. Fibrations are closed under composition.
3. The pullback of a fibration along any other map exists and is again a fibration.
4. The pullback of an acyclic fibration along any other map is again an acyclic fibration.
5. Weak equivalences satisfy 2-out-of-6.
6. Isomorphisms are acyclic fibrations and every acyclic fibration has a section.
7. Every object B has at least one path object.

(This is similar to Brown's notion of a *category of fibrant objects*, but strengthens it in two ways: we have 2-out-of-6 for weak equivalences instead of 2-out-of-3 and we demand that acyclic fibrations have sections.)

Examples

Examples

1. The fibrant objects in a model category in which every object is cofibrant (simplicial sets, groupoids, topological spaces, ...).
2. The effective topos (Van Oosten).
3. The syntactic category associated to Martin-Löf type theory (Avigad-Kapulkin-Lumsdaine).
4. The syntactic category remains an example even when the computation rule for J only holds propositionally (BvdB: see arXiv:1604.06001).
5. Any category with finite limits (all maps fibrations, only isomorphisms are weak equivalences).

First basic facts about path categories

- Every map $f : Y \to X$ factors as a weak equivalence followed by a fibration:

$$\begin{array}{ccccccc} Y & \xrightarrow{(1,rf)} & P_f & \xrightarrow{p_2} & PX & \xrightarrow{t} & X \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle s} & & \\ & & Y & \xrightarrow[f]{} & X & & \end{array}$$

- This means that if $f : Y \to X$ is a fibration, then we can factor $Y \to Y \times_X Y$ as

$$Y \longrightarrow P_X(Y) \longrightarrow Y \times_X Y,$$

  where the first is a weak equivalence and the second a fibration.
- Corollary: Let $\mathcal{C}(X)$ be the full subcategory of $\mathcal{C}/X$ whose objects are fibrations. Then $\mathcal{C}(X)$ is again a path category.

Homotopy in a path category

If $f, g : Y \to X$ are two parallel maps, then we say that f and g are *homotopic* and write $f \simeq g$ if there is a map $H : Y \to PX$ making

$$\begin{array}{ccc} & & PX \\ & \overset{H}{\nearrow} & \downarrow{\scriptstyle (s,t)} \\ Y & \xrightarrow[(f,g)]{} & X \times X \end{array}$$

commute.

Theorem

The homotopy relation $\simeq$ is a congruence on $\mathcal{C}$.

The quotient is the *homotopy category* of $\mathcal{C}$. A map which becomes an isomorphism in the homotopy category is called a *homotopy equivalence*.

Theorem

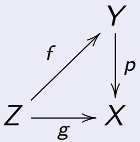
The weak equivalences and homotopy equivalences coincide in a path category.

A useful property

A useful property of path categories is the following:

Proposition

Suppose p is a fibration fitting into a triangle



which commutes up to homotopy. Then there is a map $f' : Z \to Y$, homotopic to f, such that for f' the triangle commutes strictly with $pf' = g$.

Homotopy equivalence relations

How do we modify the exact completion for path categories?

Homotopy equivalence relaton

A fibration $\rho = (\rho_1, \rho_2) : R \to X \times X$ is a *homotopy equivalence relation* if there are

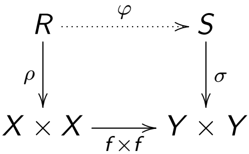
1. a map $k : X \to R$ such that $\rho k = \Delta_X$,
2. a map $l : R \to R$ such that $\rho_1 l = \rho_2$ and $\rho_2 l = \rho_1$, and
3. a map $m : R \times_X R \to R$ such that $\rho_1 m = \rho_1$ and $\rho_2 m = \rho_2$.

Proposition

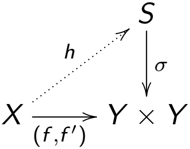
If PX is a path object on X, then $(s, t) : PX \to X \times X$ is a homotopy equivalence relation on X. Indeed, it is the "smallest" such.

Morphisms

We will organise the homotopy equivalence relations into a category. So, if $(X, \rho : R \to X \times X)$ and $(Y, \sigma : S \to Y \times Y)$ are two homotopy equivalence relations, then a morphism $f : (X, R) \to (Y, S)$ is an equivalence class of maps $f : X \to Y$ for which there is a "tracking" φ making



commute. We will regard two such maps $f, f' : X \to Y$ as equivalent if there is a map $H : X \to S$ making



commute.

Homotopy exact completion

This results in a category which we will call the *homotopy exact completion* of $\mathcal{C}$ and denote by $\mathrm{Hex}(\mathcal{C})$. Note that there is again a functor $Y : \mathcal{C} \to \mathrm{Hex}(\mathcal{C})$, this time obtained by sending X to (X, PX).

Theorem

$\mathrm{Hex}(\mathcal{C})$ is exact.

Interesting (and useful!) fact:

Theorem

$\mathrm{Hex}(\mathcal{C})$ is the homotopy category of a path category $\mathrm{Ex}(\mathcal{C})$.

Exact completion for path categories

The objects of $\mathrm{Ex}(\mathcal{C})$ are the same as those of $\mathrm{Hex}(\mathcal{C})$, that is, homotopy equivalence relations. Morphisms $f : (X, R) \to (Y, S)$ are maps $f : X \to Y$ for which there is a "tracking" φ making

$$\begin{array}{ccc} R & \overset{\varphi}{\dashrightarrow} & S \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\sigma} \\ X \times X & \xrightarrow[f \times f]{} & Y \times Y \end{array}$$

commute. Define:

- $f \sim f'$ for two such maps if there exists $H : X \to S$ such that $(f, f') = \sigma H$.
- Such a map f is a *weak equivalence* if there is a morphism $g : (Y, S) \to (X, R)$ such that $fg \sim 1$ and $gf \sim 1$.
- Such a map f is a *fibration* if $f : X \to Y$ is a fibration in $\mathcal{C}$ and there is a map $L : X \times_Y S \to R$ such that $\rho_1 L = p_1$ and $f \rho_2 L = \sigma_2 p_2$.

Homotopy exponentials

Theorem

The category $\mathrm{Ex}(\mathcal{C})$ is a fibration category whose homotopy relation is given by $\sim$. Therefore its homotopy category is precisely $\mathrm{Hex}(\mathcal{C})$.

Homotopy exponentials

Suppose X and Y are two objects in a path category $\mathcal{C}$. A *homotopy exponential* is an object X^Y together with a map $\mathrm{ev} : X^Y \times Y \to X$ such that for any map $f : A \times Y \to X$ there is a map $F : A \to X^Y$, unique up to homotopy, such that

$$\begin{array}{ccc} X^Y \times Y & \xrightarrow{\mathrm{ev}} & X \\ {\scriptstyle F \times 1}\uparrow & \nearrow_{f} & \\ A \times Y & & \end{array}$$

commutes up to homotopy. If we drop the requirement that F is unique up to homotopy, then we call X^Y a *weak homotopy exponential*.

Homotopy Π-types

Homotopy exponentials

Suppose $f : X \to J$ and $\alpha : J \to I$ are two fibrations. A *homotopy* Π*-type* of f along α is a fibration $\Pi_\alpha(f) \to I$ together with a map $\mathrm{ev} : \alpha^*\Pi_\alpha(f) \to f$ in $\mathcal{C}(J)$, such that: for any map $g : Y \to I$ and $s : \alpha^* Y \to X$ over J there is a map $S : Y \to \Pi_\alpha(X)$ over I and unique up to fibrewise homotopy over I, such that $s \simeq_J \mathrm{ev} \circ \alpha^* S$. If we drop the requirement that S is unique up to fibrewise homotopy, then we call $\Pi_\alpha(f)$ a *weak homotopy* Π*-type*.

Sample theorem

If $\mathcal{C}$ is a path category with weak homotopy Π-types, then $\mathrm{Ex}(\mathcal{C})$ has homotopy Π-types and $\mathrm{Hex}(\mathcal{C})$ is locally cartesian closed.

Interesting directions

- Effective topos as the homotopy category of a path category?
- Is there also a homotopy regular completion?
- What are $\mathrm{Hex}(\mathrm{Top})$ and $\mathrm{Ex}(\mathrm{Top})$? Is there a homotopic version of Scott's equilogical spaces?